2-D FFT ALGORITHM FOR FRANK SEQUENCE PROCESSING

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The paper presents a new fast algorithm to calculate periodic auto- and crosscorrelation functions of Frank sequences. It is based on a 2-dimensional representation of the sequences and a combination of the spectral and direct correlation function calculation methods. In this case the computational accuracy is enhanced and the number of the required complex operations is reduced twice.

INTRODUCTION

For the recent 30 years intensive research activities have been conducted globally in the field of the theory and practical application of spread-spectrum signals. Radio communication, navigation, radar, information protection, wireless computer network, spectrometry systems – this is far from being a complete list of broadband signals applications. Their potential application in hydroacoustics is of special interest.

Multipath and nonstationary nature of the hydroacoustic channel makes solution of the problems related to data transmission noise immunity, navigation and detection of various underwater objects rather complicated. Correlation and statistic properties of broadband signals allow minimizing the impact of negative factors. First of all we speak about a capability of fast and accurate communication channel measurements and creation of adaptive hydroacoustic systems. Owing to ideal correlation properties it is possible to accumulate energy of signals, arriving by individual beams. In the location systems there appears a capability of increasing the detection range of underwater objects, of improving the equipment resolution and reduction of demasking emissions.

One of the techniques to extend the signal spectrum is based on using pseudo-random sequences. The best known, well studied and widely applied are binary sequences, such as M-sequences, Gold, GMW, Kassamy, and Golay sequences. Efficient digital processing algorithms were developed for these sequences. Complex sequences have been acquiring ever greater attention of scientists recently. They are employed to generate multi-phase signals, characterized by ideal correlation properties. Among them Frank sequences are distinguished. They are easy to generate and have a minimal number of various phases. As of today fast

processing algorithms for Frank sequences have not been sufficiently studied. In what follows we shall describe a correlation algorithm that requires lesser number of arithmetic operations comparing to a classical correlation processing algorithm based on the fast Fourier transform.

1. FRANK SEQUENCES

Frank sequences [1] are multi-phase sequences having periodic autocorrelation function (PACF). They are determined for the case when their length is $n = m^2$, where m is any integer, and they are formed by means of DFT matrix of the dimension $m \times m$ Let $\{a_i\} = (a_0, a_1, ..., a_{n-1})$ be a Frank sequence of the length n. We shall now form the matrix $\mathbf{F}$ pursuing the following rule:

$$\mathbf{F} = \left[w^{i_1 i_2}\right], w = \exp\left(-j\frac{2\pi}{m}\right), \; j = \sqrt{-1}, \tag{1}$$

where: $i_1 = 0,1,...,m-1$ stands for the line number and $i_2 = 0,1,...,m-1$ is the column number in the matrix.

As a result we have

$$\mathbf{F} = \begin{bmatrix} w^0 & w^0 & \cdots & w^0 \\ w^0 & w^1 & \cdots & w^{m-1} \\ \vdots & \vdots & \ddots & \vdots \\ w^0 & w^{m-1} & \cdots & w^1 \end{bmatrix}. \tag{2}$$

The Frank sequence is formed by means of sequential reading the line of the matrix $\mathbf{F}$. In general it appears as follows:

$$\{a_i\} = [w^0, w^0, ..., w^0, w^0, w^1, ..., w^{m-1}, ..., w^0, w^{m-1}, ..., w^1]. \tag{3}$$

Its each symbol may be calculated with the help of an analytic form

$$a_i = w^{l_i} \tag{4}$$

where l_i stands for the parameter dependent on the symbol number in the sequence $\{a_i\}$ and determining its actual values of $l_i = i_1 i_2$, and i_2 is interrelated with i_1 through the expression $i = i_1 + m i_2$.

According to the definition the amplitude value of w is 1, hence, $|a_i| = 1$ for all values of i. Elements of $\{a_i\}$ are the elements of the matrix $\mathbf{F}$, therefore its alphabet consists of the multitude such as $w^0, w^1, ..., w^{m-1}$, and a signal, generated on the basis of the Frank sequence shall have only m various phases. It is well-known that the DFT spectrum $\{a_i\}$ shall be uniform, for this reason its PACF shall be ideal.

Sequences derived from the modified matrix $\mathbf{F}$ represent generalization of the Frank sequences [2, 3]:

$$\mathbf{F}_{k,l=}\mathbf{F}\mathbf{P}_k diag\mathbf{D}_l \tag{5}$$

where $\mathbf{P}_k$ is the permutation matrix of the dimension $m \times m$, $k = 1,...,K$, $diag\mathbf{D}_l$ is formed by the vector $\mathbf{D} = [1, w^{l_1}, w^{l_2}, ..., w^{l_{m-1}}]$, $l_{i_1} = 0,...,m-1$, $l = 1,...,L$.

Parameters K and L determine the general number of various Frank sequences, that is $N = KL$ and can be calculated using formulas

$$K=(m-1)!,\quad L=m^{m-1}. \tag{6}$$

Hence, $N=m^{m-2}m!$.

As an example we shall form the Frank sequence for $n=16,$ using the DFT matrix of the $m\times m=4\times 4$ dimension, exchange and diagonal matrices:

$$\mathbf{F}=\begin{bmatrix}1 & 1 & 1 & 1\\ 1 & w^1 & w^2 & w^3\\ 1 & w^2 & 1 & w^2\\ 1 & w^3 & w^2 & w^1\end{bmatrix},\mathbf{P}_k=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ 0 & 1 & 0 & 0\end{bmatrix},$$

$$diag\mathbf{D}_l=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & w^1 & 0 & 0\\ 0 & 0 & w^2 & 0\\ 0 & 0 & 0 & w^3\end{bmatrix}. \tag{7}$$

Then, taking into consideration, that $w=\exp\left(-j\frac{\pi}{2}\right)=-j$ we shall obtain

$$\mathbf{F}_{k,m}=\begin{bmatrix}1 & w & w^2 & w^3\\ 1 & 1 & w^3 & w\\ 1 & w^3 & 1 & w^3\\ 1 & w^2 & w & w\end{bmatrix}=\begin{bmatrix}1 & -j & -1 & j\\ 1 & 1 & j & -1\\ 1 & j & 1 & j\\ 1 & -1 & -j & -j\end{bmatrix} \tag{8}$$

and the Frank sequence

$$\left\{a_i^{k,l}\right\}=[1,-j,-1,\ j,1,1,\ j,-1,1,\ j,1,1,-1,-j,-j]. \tag{9}$$

The calculation results for PACF of the sequence obtained are given in Figures below.

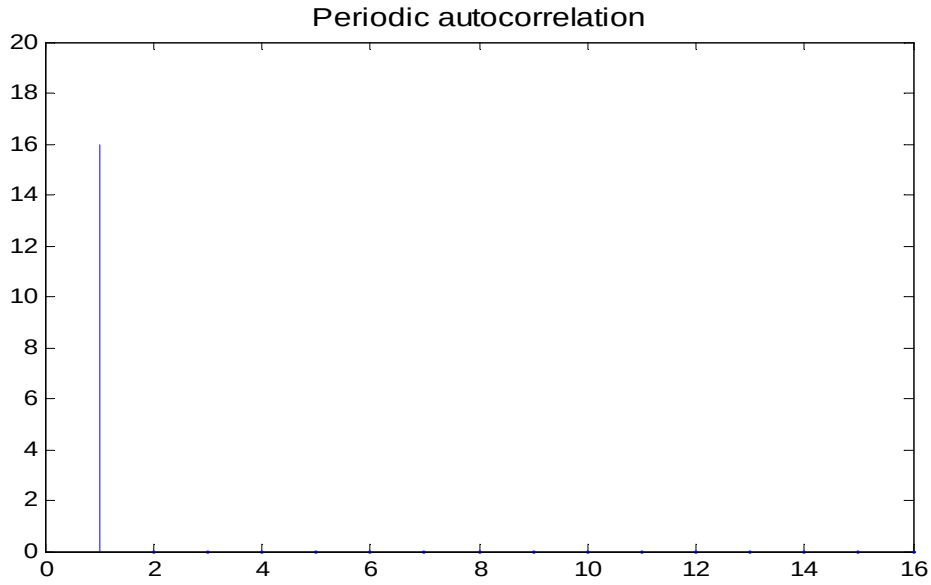

Fig.1 Periodic autocorrelation function for the formed Frank sequence

2. COMPUTATIONAL COMPLEXITY OF CORRELATION FUNCTIONS

In case of digital signal processing the complexity of the algorithm, usually estimated by the number of arithmetic operations required for its implementation, is of crucial significance.

Algorithm 1.

Algorithm 1 is the most labor-intensive. The correlation function is calculated directly by multiplying the vector **a**, formed by the symbols of the sequence $\{a_i\}$ by the matrix **B**, composed of cyclic shift symbols of the sequence $\{b_i\}$. The length of both the sequences is n, and the dimension of the **B** matrix is $n \times n$. Then the correlation vector **R** may be determined by the below formula

$$\mathbf{R} = \mathbf{Ba}. \tag{10}$$

To multiply the vector by the matrix it is required to carry out n^2 of multiplications and $n(n-1)$ summations. If the correlation function is calculated for the Frank sequences, all the operations are complex.

Algorithm 2.

Processing of the correlation function in the spectral domain saves a great number of arithmetic operations. Let the matrix **F** of the Fourier transform have the dimension $n \times n$

$$\mathbf{F} = \left[w_1^{i_1 i_2} \right], \; w_1 = \exp\left(-j \frac{2\pi}{n} \right) \tag{11}$$

where: $i_1 = 0,1,...,n-1$ stands for the line number and $i_2 = 0,1,...,n-1$ is the column number in the matrix.

Then

$$\mathbf{R} = \mathbf{F}^{-1}\left((\mathbf{Fa}) \times (\mathbf{Fb})^T \right) \tag{12}$$

here $\mathbf{F}^{-1}$ is the matrix of the Fourier inverse, $(\mathbf{X})^*$ represents the matrix conjugated with the matrix **X**, and $\times$ stands for the symbol of symbol-by-symbol vector multiplication.

As it follows from the expression, the Fourier transform needs to be calculated three times and n complex multiplication be made. Usually **Fb** is calculated in advance, only once, therefore correspondent operations of estimating the complexity shall further be neglected. The complexity of the algorithm is very much dependent on the sequence length and shall be minimal if $n = 2^k$. In this case it will require no more than $2nk$ of complex summations and not more than $2n(k+1)$ multiplications of complex numbers.

The block diagram of the algorithm is shown in Fig. 2.

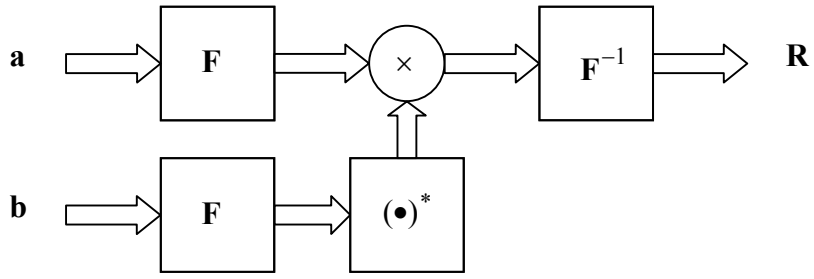

Fig.2 Block diagram of algorithm 2

3. MAIN ALGORITHM

Algorithm 3 (2-dimensional algorithm).

This algorithm is a kind of the spectral method (Algorithm 2), which employs the Fourier transform of the dimension $m \times m$ ($m = \sqrt{n}$) and a 2-dimensional representation of the sequences $\{a_i\}$ and $\{b_i\}$ [4]. Let $\mathbf{F}_1$ be an $m \times m$-dimensional Fourier transform matrix. We shall now introduce an additional matrix $\mathbf{V}$ of the dimension $m \times m$

$$\mathbf{V} = \left[w_1^{i_1 i_2} \right], \; w_1 = \exp\left(-j \frac{2\pi}{n} \right), \tag{13}$$

where: $i_1 = 0,1,...,m-1$ is the line number and $i_2 = 0,1,...,m-1$ is the column number in the matrix $\mathbf{V}$.

Now we shall determine the operation of direct multiplication of matrices $\mathbf{E} = \mathbf{C} \times \mathbf{D}$ using the following method:

$$\text{if } \mathbf{C} = \begin{bmatrix} c_0 & c_2 \\ c_1 & c_3 \end{bmatrix}, \text{ and } \mathbf{D} = \begin{bmatrix} d_0 & d_2 \\ d_1 & d_3 \end{bmatrix},$$

$$\text{then } \mathbf{E} = \mathbf{CD} = \begin{bmatrix} c_0 d_0 & c_2 d_2 \\ c_1 d_1 & c_3 d_3 \end{bmatrix}.$$

We shall rewrite the sequences $\{a_i\}$ and $\{b_i\}$ in a form of $m \times m$-dimensional matrices

$$\mathbf{A} = \begin{bmatrix} a_0 & a_1 & \cdots & a_{m-1} \\ a_m & a_{m+1} & \cdots & a_{2m-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m(m-1)} & a_{m(m-1)+1} & \cdots & a_{n-1} \end{bmatrix}$$

$$\text{and } \mathbf{B} = \begin{bmatrix} b_0 & b_1 & \cdots & b_{m-1} \\ b_m & b_{m+1} & \cdots & b_{2m-1} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m(m-1)} & b_{m(m-1)+1} & \cdots & b_{n-1} \end{bmatrix}$$

Then we shall calculate intermediate matrices $\mathbf{W}_a$ and $\mathbf{W}_b$, DFT spectral coefficients of the sequences $\{a_i\}$ and $\{b_i\}$ being the elements thereof

$$\mathbf{W}_a = \mathbf{F}_1 \left(\mathbf{V} \times \mathbf{F}_1 \mathbf{A} \right)^T, \; \mathbf{W}_b = \mathbf{F}_1 \left(\mathbf{V} \times \mathbf{F}_1 \mathbf{B} \right)^T. \tag{14}$$

Using the 2-dimensional inverse DFT we shall obtain a correlation matrix

$$\mathbf{R}_1 = \mathbf{F}_1^{-1} \left(\mathbf{V}^{-1} \times \left(\mathbf{F}_1^{-1} \left(\mathbf{W}_a \times \mathbf{W}_b^* \right) \right) \right)^T. \tag{15}$$

It is well-known that there is the following interrelation between $\mathbf{R}_1$ and $\mathbf{R}$

$$\mathbf{R} = \left[r_0, r_1, ..., r_{n-1} \right], \tag{16}$$

$$\mathbf{R}_1 = \begin{bmatrix} r_0 & r_1 & \cdots & r_{m-1} \\ r_m & r_{m+1} & \cdots & r_{2m-1} \\ \vdots & \vdots & \ddots & \vdots \\ r_{m(m-1)} & r_{m(m-1)+1} & \cdots & r_{n-1} \end{bmatrix}. \tag{17}$$

That is $\mathbf{R}_1$ and $\mathbf{R}$, are 2-dimensional and one-dimensional representations of one and the same multitude of values of the mutual correlation function for the sequences $\{a_i\}$, $\{b_i\}$ correspondingly.

To analyze the algorithm we shall now represent it in a parallel form

$$\mathbf{w}_a = \mathbf{TF}_t\mathbf{T}\left(\mathbf{v}\times\mathbf{F}_t\mathbf{Ta}\right),\ \mathbf{w}_b = \mathbf{TF}_t\mathbf{T}\left(\mathbf{v}\times\mathbf{F}_t\mathbf{Tb}\right),\ \mathbf{R} = \mathbf{TF}_t^{-1}\mathbf{T}\left(\mathbf{v}^{-1}\times\left(\mathbf{F}_t^{-1}\mathbf{T}\left(\mathbf{w}_a\times\mathbf{w}_b^*\right)\right)\right), \quad (18)$$

where $\mathbf{T}$ stands for the exchange matrix, v is the column vector, formed by the sequential account of columns in the matrix $\mathbf{V}$, $\mathbf{F}_t = \mathbf{F}_1\otimes\mathbf{E}, \mathbf{F}_t^{-1} = \mathbf{F}_1^{-1}\otimes\mathbf{E}$, $\mathbf{E}$ represents the unitary matrix of the order m, and symbol $\otimes$ means Kronecker tensor product of matrices.

The block diagram of algorithm 3 in a parallel form as per (14), (15) and (18) is shown in Fig. 3.

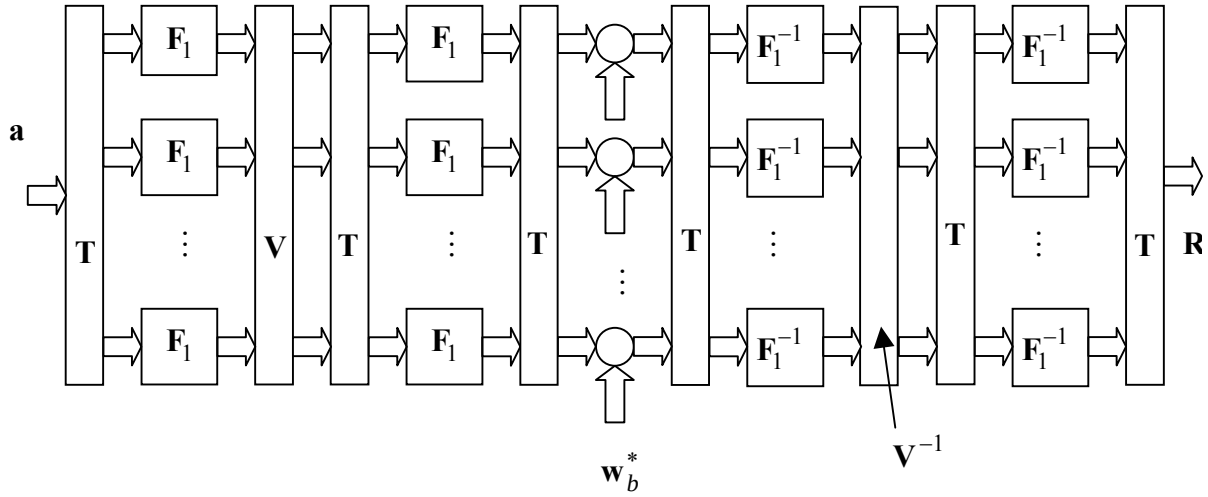

Fig.3 Block diagram of algorithm 3

Immediate calculation of the number of arithmetic operations suggests, that in terms of computational complexity algorithms 2 and 3 are equivalent.

Algorithm 4.

For the optimization purposes of Algorithm 3 we shall now consider its central part, where calculations are to be made according to formulas (18). We shall designate $\mathbf{A}_f = \mathbf{T}\left(\mathbf{v}\times\mathbf{F}_t\mathbf{Ta}\right)$, $\mathbf{B}_f = \mathbf{T}\left(\mathbf{v}\times\mathbf{F}_t\mathbf{Tb}\right)$, $\mathbf{w}_a = \mathbf{TF}_t\mathbf{A}_f$, $\mathbf{w}_b = \mathbf{TF}_t\mathbf{B}_f$ and then $\mathbf{R}_f = \mathbf{F}_t^{-1}\mathbf{T}\left(\mathbf{w}_a\times\mathbf{w}_b^*\right)$, $\mathbf{R} = \mathbf{TF}_t^{-1}\mathbf{T}\left(\mathbf{v}^{-1}\times\mathbf{R}_f\right)$.

The inner part is composed of the following operations

$$\mathbf{w}_a = \mathbf{TF}_t\mathbf{A}_f,\ \mathbf{w}_b = \mathbf{TF}_t\mathbf{B}_f \quad (19)$$

$$\mathbf{R}_f = \mathbf{F}_t^{-1}\mathbf{T}\left(\mathbf{w}_a\times\mathbf{w}_b^*\right). \quad (20)$$

These expressions describe computation of the mutual correlation function $\mathbf{R}_f$ for the sequences $\mathbf{w}_a$ and $\mathbf{w}_b$ by means of the discrete Fourier transform. Fig. 4 plots the block diagram of Algorithm 3 in a parallel form as per (19) and (20).

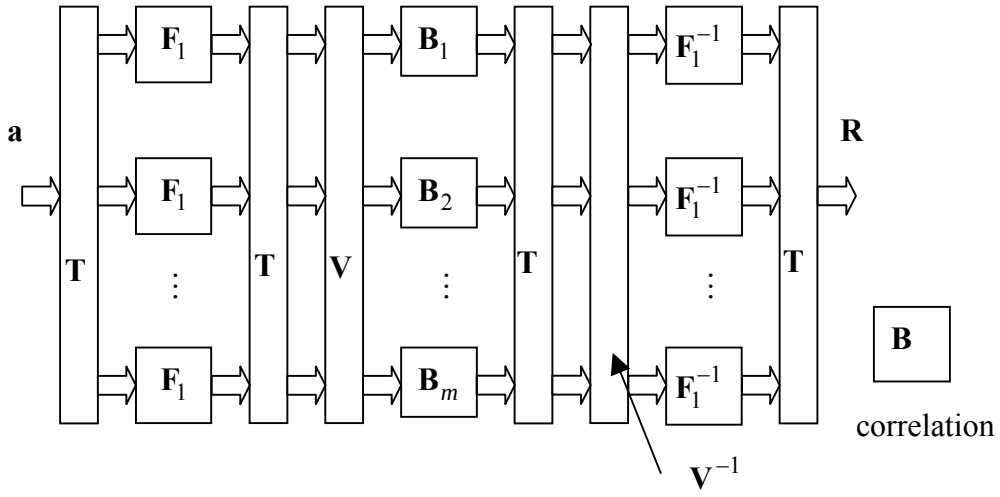

Fig.4 Block diagram of the modified Algorithm 3

$\mathbf{R}_f$ may be calculated by direct multiplication of the vector $\mathbf{w}_a$ by the matrix $\mathbf{B}$ composed of m cyclic shifts of the sequence $\mathbf{w}_b$

$$\mathbf{R} = \mathbf{B}\mathbf{w}_a . \tag{21}$$

In a general case this will require huge computational resources, if compared to Algorithm 2. If the sequence $\mathbf{w}_a$ has many zeros, this method is likely to be much simpler.

It follows from the Fran sequence algorithm, that the vector $\mathbf{A}$ is composed out of n divergent lines of an orthogonal matrix $\mathbf{F}$. Therefore the product $\mathbf{F}_t\mathbf{A}_f$ represents a vector, having only m nonzero values. And each fragment of $\mathbf{w}_a$ contains only one nonzero value. Hence, calculating (21) will involve a cyclic shift of each of m sequence fragments, and then multiplying it by m.

For the purposes of further simplification of the algorithm we should merge multiplications $\mathbf{w}_b$ by $\mathbf{v}$, $\mathbf{v}^{-1}$ and $\mathbf{w}_a^*$ and perform only one process of multiplying by an equivalent matrix $\mathbf{P}$. As a result we shall receive algorithm 4, its block diagram shown in Fig. 5.

The 2-dimensional algorithm obtained is constituted of the following stages:

1. Transforming the vector (the sequence) into a 2-D $m \times m$-dimensional matrix
2. Calculating the DFT for the matrix columns;
3. Transposing the DFT spectrum matrix;
4. Performing cyclic shifts in columns;
5. Multiplying by the equivalent matrix;
6. Matrix transposition;
7. Computing the inverse DFT for the columns;
8. Transforming the correlation matrix into the vector (sequence).

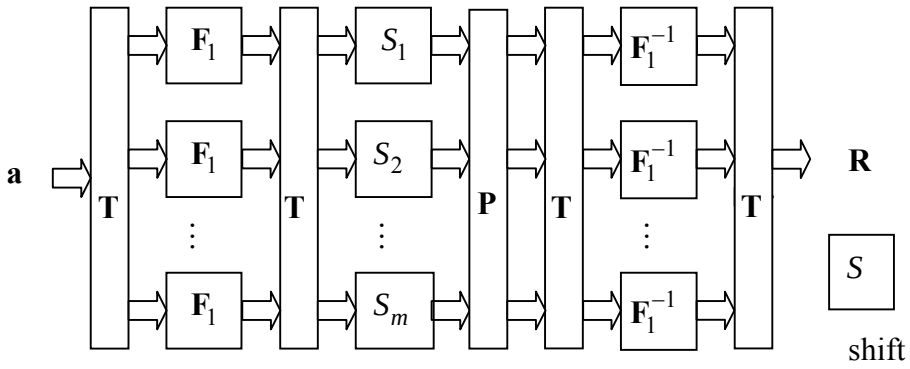

Fig.5 Block diagram of Algorithm 4

Immediate calculation suggests that for implementing Algorithm 4 it is required to count $2m$ DFT of $m \times m$ dimension and to perform no more than n intermediate complex multiplications. In such a way it is possible to reach a double-ply reduction in the number of the operations comparing to algorithm 2.

4. SIMULATION

Simulation of the algorithm developed to calculate the Frank sequence correlation functions was performed in the MatLab environment. The objective of this simulation was to demonstrate the operability of the algorithm, to show that a new algorithm is by no means poorer in terms of noise immunity if compared to the conventional correlation algorithms. In addition it enabled to solve an applied task of automating the algorithm synthesis process for arbitrary-length sequences and random exchange and diagonal matrices, modulating the sequence.

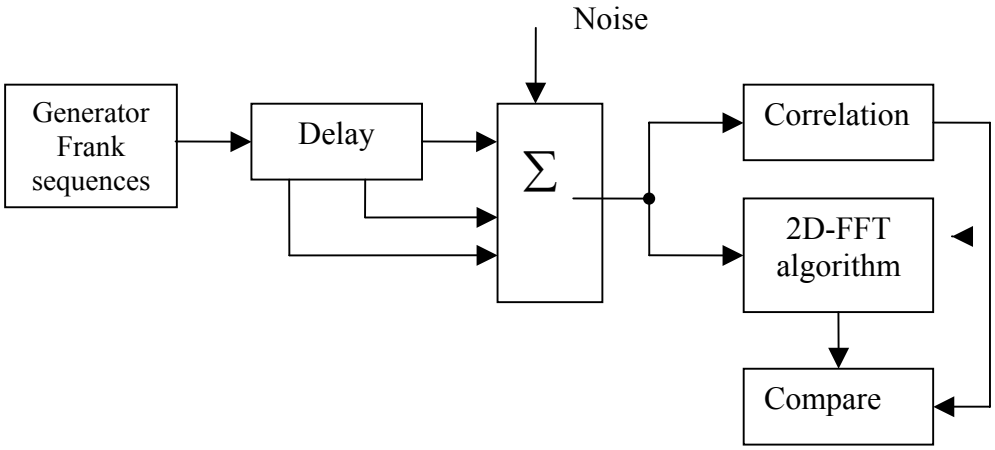

Fig.6 Software model

The software model contains the Frank sequence generator and two processing paths. The first path contains a software processor that implements the conventional PACF computation scheme – algorithm 2. The second path is programmed to realize a new 2-D

algorithm – FFT (algorithm 4). The processing results from the two paths are compared in a comparator.

To make it visually clearer the algorithm models are equipped with a simulator unit of the interference multi-beam channel. A three-beam channel model is employed for the simplicity purposes. A relative signal delay from different beams is divisible into the length of a single symbol in the Frank sequence. The adder performs weighing up of signals from different beams and of noise.

Fig. 7 reviews the calculation results for the cross-correlation function of a signal, composed of three Frank sequences 1024 in length, with relative delays of 150, 200, 500 symbols and also of a non-delayed Frank sequence. Fig. 8 contains the results of similar calculations in presence of noise.

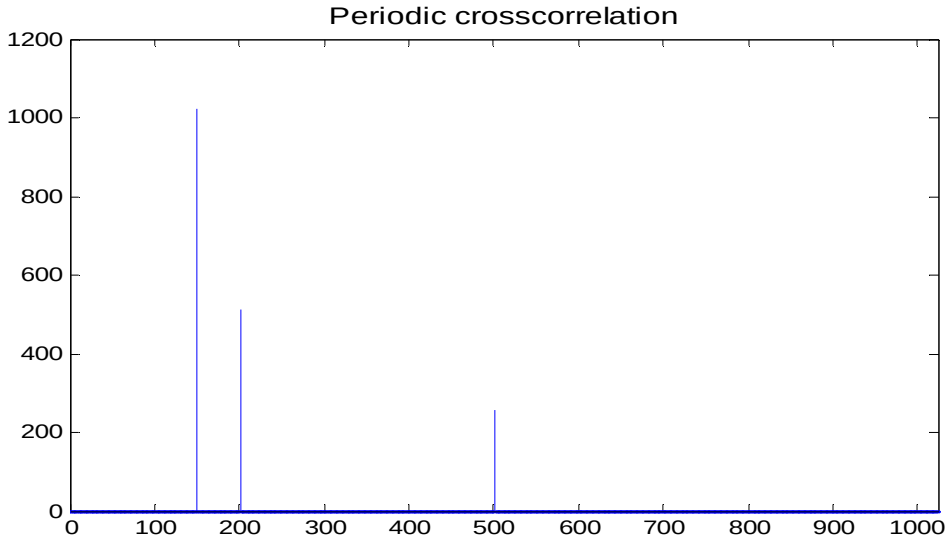

Fig.7 Results of the mutual correlation function calculation for an noise-free signal

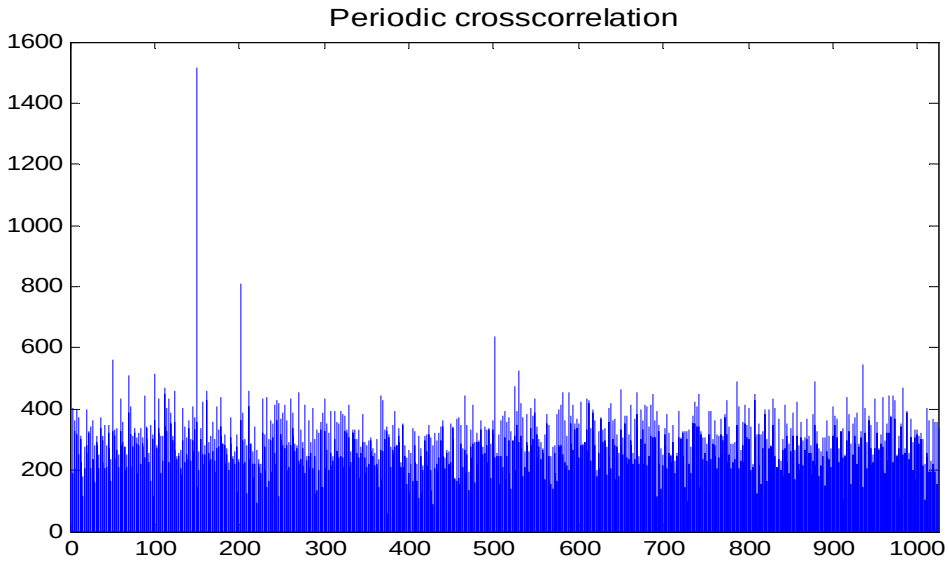

Fig.8 Mutual correlation function of a signal in presence of noise

Simulation proved that both algorithms are completely equivalent and have identical noise immunity.

5. CONCLUSIONS

The paper analyzes computation techniques of Frank sequences correlation functions. It has been suggested to employ a 2D fast Fourier transform algorithm for the spectral method, thus eliminating excessive calculations. With the noise immunity preserved a new algorithm is twice as superior as the conventional one. In addition the computational accuracy is improved owing to a reduction in the number of complex multiplications. Its practical implementation does not seem difficult. Along with hydroacoustics the algorithm described is well-suited for application in wireless broadband communication systems and channel division communication systems.

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