THE MATHIEU FUNCTIONS APPLIED TO SOME PROBLEMS IN UNDERWATER ACOUSTICS

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The purpose of this paper is to present examples of the application of the Mathieu functions to solving problems in the field of acoustics, mainly underwater acoustics.
The Mathieu functions have application to aspects of waves related to ellipses and elliptical cylinders: elliptical membrane vibration, vibration of water in elliptical containers, etc. The reason why theses functions have attracted little attention so far is mainly the complexity of the issues they involve. Among all the other functions occurring during the separation of the variables in a wave equation (with partial derivatives) in various co-ordinate systems, the Mathieu functions were the first non-hypergeometric ones. Hence, there are difficulties in the theory of these functions, as well as the calculations involved.
The reflection of a sound-wave is studied at an inhomogeneous layer with parallel surfaces separating two homogeneous semi-infinite media having different indices.

INTRODUCTION

Special functions used in the theoretical acoustics were introduced as a result of the search for the solutions of practical problems. In the process of mathematical description of an acoustical phenomena scientists apply mathematical methods, which may facilitate physical interpretation of the results obtained. Mathieu introduced certain functions, which were then given his name, in relation to the problem of vibrations of an elliptically-shaped membrane. The acoustic waves propagating in the fluid must satisfy the wave equation [7]:

$$(\nabla^2 + k^2)V(x,y) = 0 \tag{1}$$

where ∇^2 is two-dimensional Laplace operator, and k is the wave number.
Wave equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + k^2 V = 0 \tag{2}$$

after moving on to elliptical co-ordinates (ξ,η), assumes the form [8, 9, 10]:

$$\frac{\partial^2 V}{\partial \xi^2}+\frac{\partial^2 V}{\partial \eta^2}+k'^2\left(ch^2\eta-\cos^2\xi\right)V=0 \tag{3}$$

where:

$k'=ke$, e – ellipsis excentricity, $x+iy=e\cos(\xi+i\eta)$ and $\eta=const$ equation of concentric ellipses, $\xi=const$ equation of concentric hyperbolas.

A solution of equation (3) may be written in the form:

$$V(\xi,\eta)=\Xi(\xi)\cdot H(\eta) \tag{4}$$

Substituting (4) to equation (3), we will obtain, (after separating the variables):

$$\frac{1}{\Xi}\frac{d^2\Xi}{d\xi^2}+k'^2\cosh^2\xi=-\frac{1}{H}\frac{d^2H}{d\eta^2}+k'^2\cos^2\eta \tag{5}$$

After equating to the constant of both sides, we will obtain the following Mathieu equation:

$$\frac{d^2\Xi}{d\xi^2}+\left(a-k'^2\cos^2\xi\right)\Xi=0 \tag{6}$$

and modified Mathieu equation:

$$\frac{d^2H}{d\eta^2}-\left(a-k'^2 ch^2\eta\right)H=0 \tag{7}$$

where: a is a constant

Ξ is a function only of variable ξ, whereas

H is a function only of variable η.

Wave equation (3) has been divided into two differential equations (6) and (7), with periodical parameters. Function H, which offers a solution of the wave equation, is a periodical function of η, (period π or 2π). The solutions of equations (6) and (7) are called the Mathieu functions [8]. Functions Ξ and H satisfy equations (8) and (9) after changing ξ and η into *x*.

Mathieu's differential equation may be written as [11]:

$$\frac{d^2y}{dx^2}+\left(b-s\cos^2 x\right)y=0 \tag{8}$$

where: *b, s* – constants (whereas the values of parameter *b* are functions of *s*)

When *x* is changed into *ix*, this equation (8) assumes the form of the modified Mathieu equation:

$$\frac{d^2y}{dx^2}-\left(b-s\cosh^2 x\right)y=0 \tag{9}$$

Their solutions may be written in various forms, depending on parameters *b* and *s*. They are four types of periodical solutions of (8) Mathieu equation [11]:

- the even solutions:

$$Se_{2r}(s,x)=\sum_{k=0}^{\infty} De_{2k}^{(2r)} \cos 2kx \qquad \text{(period } \pi \text{)} \qquad (10)$$

$$Se_{2r+1}(s,x)=\sum_{k=0}^{\infty} De_{2k+1}^{(2r+1)} \cos(2kx+1) \qquad \text{(period } 2\pi\text{)} \qquad (11)$$

- the odd solutions:

$$So_{2r}(s,x)=\sum_{k=1}^{\infty} Do_{2k}^{(2r)} \sin(2kx) \qquad \text{(period } \pi \text{)} \qquad (12)$$

$$So_{2r+1}(s,x)=\sum_{k=0}^{\infty} Do_{2k+1}^{(2r+1)} \sin[(2k+1)x] \qquad \text{(period } 2\pi \text{)} \qquad (13)$$

where *r* is the total number and De_k , Do_k are coefficients.
Three phenomena served as examples of application of the Mathieu functions to solve selected problems in acoustics: the propagation of acoustic plane wave in an acoustically non-homogenous medium, the vibrations of water in a lake of an elliptical outline and the problem of the plane acoustic wave propagation in a liquid medium in which there is an obstacle – an elliptically shaped cylinder [8].

1. THE PROPAGATION OF ACOUSTIC PLANE WAVE IN AN ACOUSTICALLY NON-HOMOGENOUS MEDIUM

Macroscopic non-homogeneity of a medium caused by a difference in temperature may be expressed in the form of the wave equation. This equation has one parameter characterizing the medium: velocity of wave propagation *c*. This velocity is a function of temperature, whereas the medium is acoustically heterogeneous.

Propagation of an acoustic plane wave in a liquid medium with a temperature gradient is found in practice quite frequently. This subject was given intense research in Great Britain, which has always been a maritime power [12]. Acoustic signalling is a significant aspect of navigation, as acoustic signals carry information on the macrostructure of the sea.

An acoustically heterogeneous medium may be conventionally divided into layers inside which the temperature is a function of (distance) height. In research of waves spreading over the sea, the medium is assumed to be layered.
The phenomenon of acoustic wave propagation in a non-homogenous medium is described by a differential equation of variable coefficients. Solving equations of this type often poses significant difficulties.

2. WAVE REFLECTION AND TRANSMISSION IN A LUQUID MEDIUM WITH A TEMTERATURE GRADIENT

Acoustic plane wave propagates in a liquid medium having variable wave number *k* (i.e. in a medium with a temperature gradient) which separates two media of wave numbers:
$k_1 = const$ and $k_2 = const$ (i.e. of constant temperatures), whereas $k_1 \neq k_2$.

The media are linked with an intermediary layer, with coefficient *k (x)* changing constantly and monotonically from k_1 to k_2.

To calculate the reflection and transmission of a wave in the above-described conditions, an equation must be written for acoustic potential *Φ(x)* (assuming that it is a one-dimensional problem) and variable wave number *k(x)*, i.e. the Helmholtz equation with a variable wave number.

The Helmholtz equation with a variable wave number may be solved using various methods. The problem has been formulated and solved by Brekhovskih [1-3]. For later on M. Jessel, [6] and G. Canevet G. Extremet [4] showed that by means of appropriate change of function, Helmholtz's variable-coefficient equation is split up into a pair of equations of the same type, but with constant coefficients in which there appear "virtual sources" containing the unknown field. By using Green's function in expressing the solution, these equations can be transformed into an integro-differential equation for the field and an integral equation for the virtual sources. The latter equation was solved numerically and hence the coefficient of reflection was deducted.

Our suggestion is follow [5]: also, a variable wave number, i.e. a *k(x)* function may be applied, which – having satisfied the aforesaid assumptions – may result in a differential equation with a known solution, i.e. the Mathieu equation [8].

3. MATHEMATICAL DESCRIPTION OF THE PROBLEM

In a liquid medium with a temperature gradient from T_1 to T_2, a plane sinusoidal wave propagates vertically to the boundary between the media, over distance *d*, with wave numbers, respectively: in medium E_1, E, E_2:

E_1; for $x<0$ $k(x) = k_1 = const,$

E ; for $0<x<d$ $k(x) \neq const$

E_2; for $x>d$ $k(x) = k_2 = const.$

Function *k(x)* changes from k_1 to k_2 constantly and monotonically. The equation for acoustic potential *Φ* and variable wave number *k(x)* obtains the form (a one-dimensional problem):

$$\frac{d}{dx}\Phi(x)+k^2(x)\Phi(x)=0 \tag{14}$$

where *Φ (x)* is an acoustic potential in medium E. Assuming that $k^2(x)$ has got the form:

$$k^2(x) = b - s\cos^2(x) \qquad \Rightarrow \qquad k(x)=\sqrt{b-s\cos^2 x} \tag{15}$$

where: *b, s* – constant, equation (14) is the Mathieu equation. Thus, known functions may be applied to solve the problem. Dependence (15) satisfies the requirements of continuity of the function and the derivatives (first and second) at the boundary between the media.

We use these dependencies:

$$k(x)=\frac{\omega}{c(x)} \tag{16}$$

where: *c(x)* is the sound velocity in the fluid. In the gaseous media we have:

$$c(x)=\sqrt{\kappa\frac{R}{\mu}T(x)}\approx 20\sqrt{T}\left[\frac{m}{s}\right] \tag{17}$$

while in the water the function *c(x)* takes the intricate form [7].
Function *k(x)* is periodical, with period π, whereas from the physical interpretation viewpoint this function is considered in the interval from *0* to $\frac{\pi}{2}$, i.e. in the interval in which it changes from k_1 to k_2. Thus:

$$k_1 = k(0) = \sqrt{b-s} \qquad \text{when} \qquad b>s \tag{18a}$$

$$k_2 = k(\frac{\pi}{2}) = \sqrt{b} \tag{18b}$$

Hence, constants *b* and *s* may be expressed with wave numbers k_1 and k_2. Consequently:

$$b = k_2^2, \quad s = k_2^2 - k_1^2 \tag{18c}$$

If wave numbers k_1 and k_2 for the media are known from the physical conditions, constants *b* and *s* appearing in equation Mathieu may be determined using (18a, 18b, 18c).

For a plane sinusoidal wave spreading vertically to the boundary between the media, its propagation is described using three equations:

- in the medium of a wave number k_1:

$$\frac{d^2\Phi_1}{dx^2} + k_1{}^2\Phi_1(x) = 0 \tag{19}$$

where k_1 and Φ_2 are the wave number and acoustic potential in medium E_1.

- in the medium of a variable wave number *k(x)*:

$$\frac{d^2\Phi(x)}{dx^2} + \left[b - s\cos^2(x)\right]\Phi(x) = 0 \tag{20}$$

where $k(x)$ and $\Phi(x)$ are the wave number and acoustic potential in medium *E*. This equation has been named the Mathieu equation and the solution may be presented using the Mathieu function.

- in the medium with a wave number k_2:

$$\frac{d^2\Phi_2}{dx^2} + k_2{}^2\Phi_2(x) = 0 \tag{21}$$

where k_2 and Φ_2 are the wave number and acoustic potential in medium E_2.

Curve (15) assumed to link the two media may be applied in the cases when it satisfactorily approximates the phenomenon in question, i.e. defines the temperature distribution in a given medium. In this way, the wave reflection coefficient may be proved to exist and be estimated and its dependence on the incident wave parameters may be studied.

4. SOLUTION OF THE PROBLEM

Solutions of equations (19) and (21) are known [5, 9], and they define the acoustic field in the media of wave numbers k_1 and k_2, respectively:

$$\Phi_1 = e^{-ik_1x} + R\cdot e^{ik_1x} \tag{22}$$

where *R* is constant,

$$\Phi_2 = Te^{-ik_2 x} \tag{23}$$

where T is constant.
A solution of the Mathieu equation adopts the form [11]:

$$\Phi = ASo_m(x,s) + BSe_m(x,s) \tag{24}$$

where So_m, Se_m are the Mathieu functions (10) and (12), m – order.
Constants R, A, B, T are determined from the boundary conditions at the boundary between the media, the continuity of acoustic pressure, and the vibration speed for $x = 0$ and $x = d$.
Consequently, for $x=0$, the boundary conditions are:

$$\rho_1 \Phi_1 = \rho \Phi, \qquad \frac{d\Phi_1}{dx} = \frac{d\Phi}{dx} \tag{25}$$

where $\rho_1 = \rho\ (0) = \rho$, and for $x = d$, analogously, they are:

$$\rho(d)\Phi = \rho_2 \Phi_2, \qquad \frac{d\Phi}{dx} = \frac{d\Phi_2}{dx} \tag{26}$$

where $\rho_2 = \rho\ (d)$. The ρ_1, ρ, and ρ_2 appearing in the boundary conditions are respective densities of the media: E_1 E, and E_2. Constant R is the reflection coefficient at the boundary between the media with wave numbers k_1 and $k(x)$ and T is the constant which is the transmission coefficient at the boundary of media E and E_2.

Finally, the boundary conditions (25) and (26) are a system of four equations with four unknowns: R, A, B, T, including the wave reflection coefficient R at the boundary between the media in the situation analysed. For $x = 0$, equations (22) and (24) lead to the results:

$$\Phi_1 = 1 + R \tag{27}$$

$$\Phi = ASo_m(0,s) + BSe_m(0,s) \tag{28}$$

and for the derivatives we obtain:

$$\left.\frac{d\Phi_1}{dx}\right|_{x=0} = -ik_1 + ik_1 R \tag{29}$$

$$\left.\frac{d\Phi}{dx}\right|_{x=0} = ASo_m'(0,s) + BSe_m'(0,s) \tag{30}$$

Analogously for $x=d$ equations (23) and (24) lead to the results:

$$\Phi_2 = Te^{-ik_2 d} \tag{31}$$

$$\Phi = ASo_m(d,s) + BSe_m(d,s) \tag{32}$$

and for the derivatives we obtain:

$$\left.\frac{d\Phi}{dx}\right|_{x=d} = ASo'_m(d,s) + BSe'_m(d,s) \tag{33}$$

$$\left.\frac{d\Phi_2}{dx}\right|_{x=d} = -ik_2 T e^{-ik_2 d} \tag{34}$$

The equation (25) and (26) adopts the form:

$$\begin{aligned} 1+R &= ASo_m(0,s) + BSe_m(0,s) \\ -ik_1 + ik_1 R &= ASo'_m(0,s) + BSe'_m(0,s) \\ ASo_m(d,s) + BSe_m(d,s) &= Te^{-ik_2 d} \\ ASo'_m(d,s) + BSe'_m(d,s) &= -ik_2 Te^{-ik_2 d} \end{aligned} \tag{35}$$

It is necessary taking into account that, for $x = 0$, $\rho_1 = \rho(0)$, while for $x = d$, $\rho(d) = \rho_2$.

The equations (35) are used to calculate constant R , i.e. the reflection coefficient at the boundary between the media with wave numbers k_1 and $k(x)$(and constant T, i.e. the transmission coefficient at the boundary between the media with wave numbers $k(x)$ and k_2).

$$|R| = \sqrt{\frac{(\vartheta + k_1 k_2 \tau)^2 + (k_2\sigma - k_1\varepsilon)^2}{(\vartheta - k_1 k_2 \tau)^2 + (k_2\sigma + k_1\varepsilon)^2}} \tag{36}$$

where:

$$\begin{aligned} \varepsilon &= So'_m(d, s)\, Se_m(0, s) \\ \vartheta &= So_m(0, s)\, Se'_m(d, s) \\ \tau &= So_m(d, s)\, Se_m(0, s) \\ \sigma &= So'_m(0, s)\, Se_m(d, s) \end{aligned} \tag{37}$$

5. DISCUSSION

The foregoing considerations are different from the ones presented in the literature [1-3, 4,6, 10] taking advantage of the same method as the one used here. The above-described method of solving the problem of wave propagation in a non-homogeneous medium, in the conditions described above, makes it possible to present the results in an analytic form using the Mathieu function. The case in question is a heterogeneous medium, of variable wave number $k(x)$, linking two homogeneous media of constant wave numbers k_1 and k_2 .

Curve (15), adopted to depict the change in the wave number depending on the temperature in a heterogeneous medium is a sample picture of temperature distribution.

6. EXAMPLES OF OTHER APPLICATIONS OF THE MATHIEU FUNCTION

1. Analysis of the vibrations of water in a lake of an elliptical outline [8].

Assumptions:

- movements of water in a lake of constant depth d is stationary all over the plane;
- dependence of the vertical movement of water particles ξ on time is $e^{i\omega t}$
- water particle movement ξ is slight.

The differential equation of movement is:

$$(\nabla^2 + k^2)\xi(x, y) = 0 \tag{38}$$

and in elliptical co-ordinates is (3), whereas the solution to be found is a combination of the Mathieu functions which represent the deformation of water surface. Function ξ represents the configuration of water surface at time t > 0. The constants occurring in the solution are determined from the initial conditions at t=0.

2. Analysis of the problem of the plane acoustic wave propagation in a liquid medium in which there is an obstacle – an elliptically shaped cylinder [8].
If there is an obstacle on the way of a spreading acoustic wave, acoustic scattering occurs. It is assumed that the obstacle, e.g. an elliptically-shaped cylinder, is many times longer than the larger axis of the ellipsis. The axis of a long elliptical cylinder of an elliptical cross-section is perpendicular to the plane of the page. The medium flows round the elliptical cylinder with velocity u in the direction forming angle v with the larger axis of the cylinder. In this case, solving the diffraction problem requires the use of elliptical co-ordinates, whereas the solution of the wave equation in such co-ordinates includes the Mathieu functions.

7. CONCLUSION

To recapitulate, the applications of the Mathieu function to solving underwater acoustic problems may be divided into two main groups:

1. Solutions of the two-dimensional wave equation written using elliptical co-ordinates;
2. Solutions of boundary conditions problems.

The majority of the applications of the Mathieu function falls into the first group of the problems related to the wave equation.

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