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DISTRIBUTIONAL SOLUTION OF A WAVE EQUATION

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Summary: The aim of this paper is to present the method of deriving the formula for a solution of the wave equation. We deal with the case when external forces are distribution-valued functions, which is e.g. the case of a quasi - point source. Also the solution is understood as a distribution-valued function.

INTRODUCTION

Originated by Sobolev [7] and Schwartz [6] theory of distributions is modern mathematical tool. Its origins reach back to physical problems where to describe some phenomena the notion of function was inadequate. It also appeared that extension of the notion of solution of a differential equation was convenient (distributional solution, weak solution).

In particular, in acoustics the Dirac distributions appear in the wave equation in the case of quasi-point sources or moving sources, [2], [3], [4].

Distributions are often treated as functions. Actually, some can (regular distributions) but most of them (e.g. Dirac delta) can merely be approximate by sequences of functions. For that reason, treating distributions as functions from mathematical point of view has only heuristic meaning.

The approach that is being presented in this paper is mathematically rigorous. In [8] it was proved the theorem on existence and uniqueness of the solution of the Cauchy problem with distribution-valued external forces and initial conditions. The theorem also gives the formula for the solution. The proof of the theorem is based on Schwartz theory [6] and new Holly theory [1] that allows us to integrate functions with distributional values rigorously.

On account of [8] we will derive the formula for the solution of the wave equation related to the acoustic pressure field of a quasi-point source.

1. NOTATIONS

We introduce the following notations (see e.g. [5]). Let $\Omega \subset \boldsymbol{R}^n$ be an open set.

$$D(\Omega) := \{\varphi \in C^\infty(\Omega): \ \ \mathrm{supp}\varphi \ \text{ is compact in } \Omega\}$$

is called the space of test functions, while

$$S := \{\varphi \in C^\infty(\mathbf{R}^n): \ \ P \cdot D^\alpha \varphi \quad \text{is bounded } \forall \alpha \in \mathbf{N}^n, P - \text{polynomial}\}$$

is the Schwartz space.

$D'(\Omega)$ denotes the space of distributions, i.e. the space of all linear continuous functionals defined on $D(\Omega)$. Distribution T is tempered ($T \in D'_{temp}$) when it has a unique continuous extension to the Schwartz space S.

A locally summable function $u: \Omega \to \mathbf{R}$ induces the functional

$$[u]: D(\Omega) \ni \varphi \mapsto \int_\Omega \varphi(x) u(x) dx \in \mathbf{R}.$$

$[u] \in D'(\Omega)$ and is called a regular distribution.
$\delta \in D'_{temp}$ is Dirac distribution and $\delta(\varphi) = \varphi(0)$ for all $\varphi \in D(\Omega)$.

2. ABSTRACT WAVE EQUATION

Consider the Cauchy problem for the abstract wave equation

$$\begin{cases} \dfrac{d^2}{dt^2} u(t) = c^2 \Delta u(t) + f(t) & \text{for a.e. } t \in J \\ u(0) = u_0 \\ u'(0) = u_1, \end{cases} \tag{1}$$

where u_0, $u_1 \in D'_{temp}$ are initial conditions and locally summable (in the sense of [1]) $f: J \to D'_{temp}$ are external forces. Constant c denotes the propagation velocity.

In that case the assumption that guarantees existence and uniqueness of the solution is of the following simple form: *the eigenvalues of the matrix*

$$\begin{pmatrix} 0 & 1 \\ -c^2 |\xi|^2 & 0 \end{pmatrix}$$

i.e. $-ic|\xi|$, $ic|\xi|$ *have real parts equal to zero for all* $\xi \in \boldsymbol{R}^n$. Therefore the solution of (1) is given by the formula (see [8])

$$u(t) = F^{-1}(\cos(tc|\xi|) \cdot \hat{u}_0) + F^{-1}\left(\frac{\sin(tc|\xi|)}{c|\xi|} \cdot \hat{u}_1\right) + \int_0^t F^{-1}\left(\frac{\sin((t-s)c|\xi|)}{c|\xi|} \cdot \hat{f}(s)\right) ds, \tag{2}$$

where $F = (\cdot)^\wedge$ denotes the Fourier transformation

$$F(T)(\varphi) = \hat{T}(\varphi) = \overline{T}(\hat{\varphi}) \quad \text{for all } T \in D'_{temp}, \varphi \in D(\Omega),$$

$$F(\psi)(\xi) = \hat{\psi}(\xi) = (2\pi)^{-\frac{n}{2}} \int_{R^n} \exp(-ix\xi)\psi(x)dx \quad \text{for all } \psi \in S, \xi \in \mathbf{R}^n.$$

3. QUASI-POINT SOURCE

Fixing $n=3$, $u_0=u_1=0$, $f(t)=q(t)\cdot\delta$, q being locally summable, in (1) we obtain inhomogeneous wave equation related to the acoustic pressure field of a quasi-point source placed at the origin, [3], [4], [2].

From (2) it follows that the solution of

$$\frac{1}{c^2}\frac{d^2}{dt^2}u(t) = \Delta u(t) + q(t)\cdot\delta \qquad \text{for a.e. } t \in J \tag{3}$$

is of the form

$$u(t) = \int_0^t q(s)F^{-1}\left(\frac{\sin((t-s)c\,|\,\xi\,|)}{c\,|\,\xi\,|}\cdot\hat{\delta}\right)ds = \left(\frac{1}{2\pi}\right)^{\frac{3}{2}}\int_0^t q(s)F^{-1}\left[\frac{\sin((t-s)c\,|\,\xi\,|)}{c\,|\,\xi\,|}\right]ds.$$

We will find inverse Fourier transform using the well known formula

$$\frac{\sin r\,|\,x\,|}{r\,|\,x\,|} = \frac{1}{4\pi}\int_{|y|=1}\exp(irxy)dy.$$

Let $\varphi \in S(\mathbf{R}^3)$. Then

$$F\left[\frac{\sin(\tau c\,|\,\xi\,|)}{c\,|\,\xi\,|}\right](\varphi) = \int_{\mathrm{R}^3}\frac{\sin(\tau c\,|\,\xi\,|)}{c\,|\,\xi\,|}\hat{\varphi}(\xi)d\xi = \sqrt{\frac{\pi}{2}}\tau\int_{|y|=1}\varphi(c\tau\, y)dy$$

and from the Schwartz theorem

$$F^{-1}\left[\frac{\sin(\tau c\,|\,\xi\,|)}{c\,|\,\xi\,|}\right](\varphi) = \frac{\tau}{4\pi}\int_{|y|=1}\varphi(c\tau\, y)dy = \frac{1}{4\pi c^2\tau}\int_{|x|=c|\tau|}\varphi(x)dx. \tag{4}$$

Consequently,

$$u(t)(\varphi) = \frac{1}{4\pi c^2}\int_0^t q(s)\frac{1}{t-s}\int_{|x|=c|t-s|}\varphi(x)dxds$$

for any $\varphi \in D(\mathbf{R}^3)$.

Looking for the spherical wave we can assume that $u(t)$ is constant on spheres, which means that $u(t) \circ \iota = u(t)$ for any ι being an isometry in $\mathbf{R}^3$. In particular, we can assume that φ depends only on $|x|$. Then

$$u(t)(\varphi) = \frac{1}{4\pi}\int_0^t \frac{1}{r}\int_{|x|=r} \varphi(|x|)dx q(t-\frac{r}{c})dr = \frac{1}{4\pi}\int_{K(0,t)} \varphi(|x|)\frac{q(t-\frac{|x|}{c})}{|x|}dx. \tag{5}$$

In that way we have obtained that

$$u(t,x) = \frac{1}{4\pi}\frac{q(t-\frac{|x|}{c})}{|x|}$$

is the solution of (3).

4. CONCLUSIONS

The obtained formula for the solution of the wave equation related to the acoustic pressure field of a quasi-point source is well known in acoustics. But this time we have derived it with no analogous or heuristic argumentation. We have used only these techniques that can be used to solve distribution-valued problems. For the mathematical details we refer the reader to [8], it also will appear in a forthcoming publication.

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