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Model Equation for the Sound Beam in Inhomogeneous Liquid

A.Perelomova[1], A.Sukhov[2]

Technical University of Gdańsk, ul. G. Narutowicza 11/12, 80-952 Gdańsk, POLAND

[1]E-mail: anpe@mif.pg.gda.pl
[2]E-mail: sukhov@mif.pg.gda.pl

A beam propagation at inhomogeneous medium is treated in parabolic approximation. For a liquid with gas bubbles a model equation which combine Khokhlov-Zabolotskaya equation with a weak dispersion was derived. Equation with new coefficients which are revised for compressible liquid is obtained. That is essential for dynamics of bubbly liquid with small volume concentrations of bubbles. A physical realistic ranges of parameters in which the derived equation is valid are specified.

1. Introduction

In many domains of investigations of acoustic field propagation through real medium, such as underwater sonar investigations, medical ultrasonic research, acoustic field scattering, effects of inhomogeneity can't be neglected. One of a popular and actual example of inhomogeneous medium is a bubble liquid.

Bubbles palaced in liquid behave as oscillators, the difference of internal (gas) and external (liquid) pressures is a reason for the boundary of bubble to move.

Small oscillations of bubbles were originally analysed by Rayleigh who assumed that surrounding liquid is incompressible and inviscid, the bubble remains spherical and surface tension is negligible. Later effects of surface tension, viscosity and heat transfer were taken in account and equation for a wide amplitude range was derived [1,2].

The equation of bubble boundary evolution is involved in general system of hydrodynamics of mixture at all

Analysis of propagation of sound waves of finite amplitude was done by Wijngaarden [3,4]. For a one-dimensional problem it is described by the famous Korteweg-de Vries (KdV) equation [4], where the nonlinear terms of second order only are taken into account and the dispersive term is a consequence of bubbles presence. A three-dimensional problem was studied by Lugovtsov [5] for incompressible liquid.

The equation for weak nonlinear sound beam propagation in homogeneous medium was obtained by Khokhlov-Zabolotskaya (KZ-equation). It is widely used in experimental research and had been solved by various numerical schemes. The solutions of KZ by perturbation methods also had been derived [7], but it is valid in a near field of source.

In the present paper a propagation of sound beam through compressible bubbly liquid is studied and the KZ approach for sound beams is applied to a system describing dynamics of bubble and water mixture.

We restrict ourselves by the following approximation. The dissipation both in liquid and gas is left out of account, there is no heat transfer between gas and liquid, also the effect of surface tension is neglected, actually it essential only for very small bubbles. All bubbles have spherical shape of same radius with uniform pressure distribution, this is argument with [1] where it is proved that velocity of pressure exualization inside the bubble is much more higher then that of temperature. The distribution of bubbles in inside liquid's volume is also uniform. The transducer is uniformly exited axial symmetrical piston with localized by zero coordinate point. The axis of propagation is x

2. Symbols

α	mass concentration of gas,
β	volume concentration of gas,
c	velocity of sound,
γ	gas specific heats ratio,
p	pressure,
ρ	mass density,
$\vec{V}$	hydrodynamic velocity with components $\{V_x, V_y, V_z\}$,
R	bubble radius,
f	linear frequency,
ω	angular frequency $\omega = 2\pi f$,
k	wave number $k = \omega / c_{mix}$,
J_n	Bessel function of the n-th order,
a	transducer radius,
r	radial variable $r = \sqrt{y^2 + z^2}$,
r_0	Rayleigh distance, $r_0 = ka^2/2$;
ε	parameter of nonlinearity,
δ	parameter of dispersion,
σ	dimensionless coordinate along acoustic axis, $\sigma = z/r_0$.
ξ	dimensionless coordinate across acoustic axis, $\xi = r/a$,
Π	dimensionless amplitude on transducer.

Values connected with liquid, gas or mixture are marked by subscript f, g, mix correspondingly. The disturbed dimensionless values are primed, equilibrium values are marked by index 0.

3. Statement of Problem

The system of equation of hydrodynamics for mixture contains equations of mass conservation and motion in differential form.

$$\frac{\partial \rho_{mix}}{\partial t} + div\left(\rho_{mix} \cdot \vec{V}\right) = 0 \tag{1}$$

$$\frac{\partial \vec{V}}{\partial t} + \left(\vec{V}\nabla\right)\vec{V} + \frac{\vec{\nabla} p_{mix}}{\rho_{mix}} = 0 \tag{2}$$

The pressure in mixture is measured in liquid phase $p_{mix} = p_f$, that corresponds to [1,3,4]. The density of mixture can be defined as

$$\frac{1}{\rho_{mix}} = \frac{\alpha}{\rho_g} + \frac{1-\alpha}{\rho_f} \tag{3}$$

It seems to be more rational to use mass concentration α instead of volume concentration β or number of bubbles per volume unit. Under assumption that gas and liquid move at the same velocity and absence of mass exchange between bubble and surrounding environment α is constant. Usage of volume concentration variable or bubbles density required an additional equation of motion for these ones.

Gas behaviour in bubble is considered to be adiabatic and mass of each single bubble is constant

$$p_g \cdot \rho_g^{-\gamma} = const \tag{4}$$

$$\rho_g \cdot R^3 = const \tag{5}$$

Bubble radius dynamics in compressible liquid is treated by equation derived by Prosperetti [1]:

$$\frac{p_g^0 p_g' - p_f^0 p_f'}{\rho_f} = \frac{3}{2}\left(\frac{\partial R}{\partial t}\right)^2 + R\frac{\partial^2 R}{\partial t^2} - \frac{1}{c_f}\left(R^2\frac{\partial^3 R}{\partial t^3} + 6R\frac{\partial R}{\partial t}\frac{\partial^2 R}{\partial t^2} + 2\left(\frac{\partial R}{\partial t}\right)^3\right) \tag{6}$$

Where $p_g^0 p_g' - p_f^0 p_f'$ is the difference between perturbation of gas and liquid pressures correspondingly.

To close the model an equation for p_g and p_f is required. For a compressible liquid an inverted virial expansion can be used:

$$\rho_f = \rho_f^0 + \frac{\rho_f^0}{A p_f^0}\left(p_f - p_f^0\right) - \frac{B\rho_f^0}{2A^3 p_f^{0\,2}}\left(p_f - p_f^0\right)^2 \tag{7}$$

Coefficients A, B are known from experimental measurements [8,9].

To treat a weak non-linear case it is useful to introduce the dimensionless variables:

$$\rho_f = \rho_f^0(1+\rho_f'),\ \rho_g = \rho_g^0(1+\rho_g'),$$
$$\rho_{mix} = \rho_{mix}^0(1+\rho_{mix}'),\ R = R_0(1+R'),$$
$$p_f = p_f^0(1+p_f'),\ p_g = p_g^0(1+p_g').$$

We would like to put attention that unlike works [1,3,4] where liquid was considered to be incompressible we consider the compressible one, so we use one equation more - the equation of liquid state (7) and one variable more - the disturbance of liquid density p_f' For convenience of analysis in this case two completely different equilibrium pressures for gas and liquid are introduced, p_g^0

being the ambient (hydrostatic) pressure, it equals to 10^5 Pa. The value p_f^0 is internal liquid pressure, for water it equals to $3 \cdot 10^8$ Pa [10,11].

The system (1-7) transforms to:

$$\frac{\partial \rho'_{mix}}{\partial t} + (1 + \rho'_{mix}) div(\vec{V}) + \vec{V} grad(\rho'_{mix}) = 0 \quad (8)$$

$$(1 + \rho'_{mix})\left(\frac{\partial \vec{V}}{\partial t} + \vec{V}\nabla\vec{V}\right) + \frac{p_f^0}{\rho_{mix}^0}\vec{\nabla}p'_f = 0 \quad (9)$$

$$\rho'_g = \frac{\alpha\rho_f^0\rho'_f\rho'_{mix} + (\rho_g^0 + \alpha\rho_f^0)\rho'_{mix} - \rho_g^0\rho'_f}{\alpha\rho_f^0 - \rho_g^0\rho'_g + (\rho_g^0 + \alpha\rho_f^0)\rho'_f} \quad (10)$$

$$R' = (1 + \rho'_g)^{-1/3} - 1 \quad (11)$$

$$p'_g = (1 + \rho'_g)^{\gamma} - 1 \quad (12)$$

$$\frac{p_g^0 p'_g - p_f^0 p'_f}{\rho_f^0 R_0^2} = \frac{3}{2}\left(\frac{\partial R'}{\partial t}\right)^2 + (1 + R')\frac{\partial^2 R'}{\partial t^2} - \\ -\frac{R_0}{c_f}\left((1+R')^2\frac{\partial^3 R'}{\partial t^3} + 6(1+R')\frac{\partial R'}{\partial t}\frac{\partial^2 R'}{\partial t^2} + 2\left(\frac{\partial R'}{\partial t}\right)^3\right) \quad (13)$$

$$\rho'_f = \frac{1}{A}p'_f - \frac{B}{2A^3}{p'_f}^2 \quad (14)$$

4. Deriving a Modelling Equation

The aim of our efforts is an equation for p'_f - the pressure perturbation in liquid or, just the same, in mixture. Following the method of pioneering work [6] we put:

$$p'_f, \rho'_{mix}, \vec{V} = F\left(\mu^2 x, \mu y, \mu z, \tau = t - z / c_{mix}\right)$$

considering $p'_f \propto \mu^2$, $\rho'_{mix} \propto \mu^2$ and saving terms up to fourth order on μ one could get the next system:

$$\frac{\partial \rho'_{mix}}{\partial \tau} - \frac{1}{c_{mix}}\frac{\partial V_x}{\partial \tau} = \\ = \mu^2\left[\frac{1}{c_{mix}}\frac{\partial(V_x\rho'_{mix})}{\partial \tau} - \frac{\partial V_x}{\partial x} - \frac{\partial V_y}{\partial y} - \frac{\partial V_z}{\partial z}\right] \quad (15)$$

$$\frac{\partial V_x}{\partial \tau} - \frac{p_f^0}{c_{mix}\rho_{mix}^0}\frac{\partial p'_f}{\partial \tau} = \\ = \mu^2\left[\frac{V_x}{c_{mix}}\frac{\partial V_x}{\partial \tau} - \rho'_{mix}\frac{\partial V_x}{\partial \tau} - \frac{p_f^0}{\rho_{mix}^0}\frac{\partial p'_f}{\partial x}\right] \quad (16)$$

$$\frac{\partial V_y}{\partial \tau} - \frac{p_f^0}{\rho_{mix}^0}\frac{\partial p'_f}{\partial y} = \\ = \mu^2\left[\frac{V_x}{c_{mix}}\frac{\partial V_y}{\partial \tau} - \rho'_{mix}\frac{\partial V_y}{\partial \tau}\right] \quad (17)$$

$$\frac{\partial V_z}{\partial \tau} - \frac{p_f^0}{\rho_{mix}^0}\frac{\partial p'_f}{\partial z} = \\ = \mu^2\left[\frac{V_x}{c_{mix}}\frac{\partial V_z}{\partial \tau} - \rho'_{mix}\frac{\partial V_z}{\partial \tau}\right] \quad (18)$$

The ρ'_{mix} can be expressed from equations (10-14) as:

$$\rho'_{mix} = C_1 p'_f - C_2\frac{\partial^2 p'_f}{\partial \tau^2} + \mu^2 C_3 {p'_f}^2 \quad (19)$$

$$C_1 = \frac{\left(\gamma p_g^0\rho_g^0 + \alpha A p_f^0\rho_f^0\right)}{A\gamma p_g^0\left(\rho_g^0 + \alpha\rho_f^0\right)} \quad (20)$$

$$C_2 = \frac{\alpha p_f^0 {\rho_f^0}^2 R_0^2}{3\gamma^2 {p_g^0}^2\left(\rho_g^0 + \alpha\rho_f^0\right)} \quad (21)$$

$$C_3 = \frac{4\alpha\rho_f^0\rho_g^0 p_f^0}{A\gamma\, p_g^0\left(\rho_g^0 + \alpha\rho_f^0\right)^2} - \\ -\frac{\left(2\alpha A\rho_f^0 + \left(\rho_g^0 + \alpha\rho_f^0\right)B\right)\rho_g^0}{A^3\gamma^2 {p_g^0}^2\left(\rho_g^0 + \alpha\rho_f^0\right)^2} - \\ -\frac{\alpha A^3\left((\gamma+1)\rho_g^0 + \alpha(\gamma-1)\rho_f^0\right)\rho_f^0 {p_f^0}^2}{A^3\gamma^2 {p_g^0}^2\left(\rho_g^0 + \alpha\rho_f^0\right)^2} \quad (22)$$

We will take in account only cases when $C_1 / C_2 \propto \mu^2$. Substituting it in (15) and (16) we get:

$$\frac{\partial V_x}{\partial \tau} + c_{mix}C_1\frac{\partial p'_f}{\partial \tau} = -c_{mix}C_2\frac{\partial^3 p'_f}{\partial \tau^3} + \\ + \mu^2 c_{mix}\left[\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} + C_3\frac{\partial {p'_f}^2}{\partial \tau} - \frac{C_1}{c_{mix}}\frac{\partial(V_x p'_f)}{\partial \tau}\right] \quad (23)$$

$$\frac{\partial V_x}{\partial \tau} - \frac{p_f^0}{c_{mix}\rho_{mix}^0}\frac{\partial p'_f}{\partial \tau} = \\ = \mu^2\left[\frac{V_x}{c_{mix}}\frac{\partial V_x}{\partial \tau} - C_1 p'_f\frac{\partial V_x}{\partial \tau} - \frac{p_f^0}{\rho_{mix}^0}\frac{\partial p'_f}{\partial x} - \right] = 0 \quad (24)$$

Under assumption $C_1 / C_2 \propto \mu^2$ we can see that first order parts of (23) and (24) differs only by constant factor, that allows to exclude the first order terms. Differentiating the result with respect to τ and excluding V_y and V_z from equation of linear order from (23, 24) and putting $\mu^2 = 1$ we get the final equation

$$\frac{\partial^2 p'_f}{\partial x \partial \tau} - \frac{\varepsilon}{2c_{mix}} \frac{\partial^2 {p'_f}^2}{\partial \tau^2} - \delta \frac{\partial^4 p'_f}{\partial \tau^4} = \frac{c_{mix}}{2} \Delta_\perp p'_f \quad (25)$$

$$\varepsilon = \frac{(2A+B)\gamma^2 \rho_g^0 {p_g^0}^2 + \alpha A^3 (\gamma+1) \rho_f^0 {p_f^0}^2}{2\gamma p_g^0 A^2 \left(\gamma p_g^0 \rho_g^0 + \alpha A p_f^0 \rho_f^0\right)} \quad (26)$$

$$\delta = \frac{\alpha A p_f^0 {\rho_f^0}^2 R_0^2}{6\gamma p_g^0 \left(\gamma p_g^0 \rho_g^0 + \alpha A p_f^0 \rho_f^0\right) c_{mix}} \quad (27)$$

The equation has "KZ" form with the dispresion term. Returning to main order of (23) and (24) expression for c_{mix} can be obtained

$$c_{mix}^2 = \frac{A p_f^0 p_g^0 \gamma \left(\rho_g^0 + \alpha \rho_f^0\right)^2}{\rho_f^0 \rho_g^0 \left(\gamma p_g^0 \rho_g^0 + \alpha A p_f^0 \rho_f^0\right)} \quad (28)$$

5. Analysis of Coefficients

In previous chapter an equation for sound beam in compressible bubbly liquid had been derived. Let's discuss the obtained results.

From one hand the "KZ" approach presuppose weak nonlineartiy of medium, it means that nonlinear constant ε should be small enough, from other hand, the numerator of ε in (26) is a sum of two terms. The first term is responsible for liquid's non-linearity and the second one for non-linearity of gas. They have a different orders depending on value of α. As an example for further analysis we takes water ($\rho_f^0 = 10^3$, $A = 7$, $B = 35$) with air ($\rho_g^0 = 1$, $\gamma = 1.4$) bubbles.

For $\alpha > 10^{-7}$ the second term is dominating over first one and compressibility of liquid is not significant. The bubbles are highly nonlinear system due to the fact that compressibility of gases is of several orders of magnitude greater then that of liquids. The basic system of equations (1-7) can be reduced to (1-6) and equation of state for liquid (7) is not needed.

For $\alpha < 10^{-8}$ both terms in (26) are significant. In this case non-linear constant also considered to be small. Hereafter we restrict ourselves by this range of mass concentration.

In order to take in account dispersion $C_1 / C_2 \propto \mu^2$ were put. It is equivalent to condition that transducer frequency ω and bubble natural frequency

$$\omega_b = \sqrt{\frac{3\gamma p_g^0}{\rho_f^0 R_0^2}}$$

are different and there is no resonance.

The sound attenuation in bubbly liquid is a function of the relation of the sound frequency to the bubble resonance frequency ω. At the problem statement the attenuation due to heat transfer and viscosity was neglected. But there is one more mechanism - scattering. Because of it sound energy redistributes from main propagation direction to other. While square of relative frequency $\omega^2 / {\omega_b}^2 << 1$ the scattering cross section is small. So in a frameworks of our problem the beam broadening can be treated as strictly diffractional. In case of resonance all neglected phenomena should be taken in account as well [12].

The value δ in general is defined by bubble radius, for given R_0 frequency of transducer for experimental observation of phenomena can be obtained. For $R_0 = 10\,\mu m$ it should be less or equal to 10^5 Hz.

In the case of bubbles absence ($\alpha = 0$) the dispersion term is equal to zero, sound velocity c_{mix} transforms into expression for linear sound velocity in pure liquid and the non-linear constant ε transforms to it's well-known form

$$c_{mix}^2 = A \frac{p_f^0}{\rho_f^0} = c_f^2, \quad \varepsilon = \frac{(2A+B)}{2A^2}$$

Last expression was obtained for weakly non-linear waves in homogeneous medium.

In the case of incompessible liquid ($A = \infty$), the nonlinear and dispersion constants can be written as

$$\varepsilon = \frac{(\gamma+1)}{2\gamma} \frac{p_f^0}{p_g^0},$$

$$\delta = \frac{\rho_f^0 R_0^2}{6 c_{mix} \gamma p_g^0}$$

This result was obtained by Wijngaarden [2], the ε differs from original formula by a constant factor only, because two different equilibrium pressures were introduced.

The mass concentration can be expressed through the initial value of volume one $\alpha = \beta_0 \dfrac{\rho_g^0}{\rho_f^0}$ and formula for linear velocity of sound in mixture also takes a known [2] form:

$$c_{mix}^2 = \frac{\gamma p_g^0}{\beta_0(1-\beta_0)\rho_f^0} \qquad (29)$$

6. Generation of Second Harmonic

The problem of second harmonics generation in field of piston source for nondispersive medium was investigated by Kunitsyn and Rudenko [7]. Here it's modification for equation (25) will be used. In general, nonsingular perturbation theory can be used for nearfield zone and moderate amplitudes of transducer. We also restrict ourselves by these assumption.

The solution of the equation (25) can be represented as Fourier series:

$$p'_f = p_1 e^{i\omega\tau} + p_2 e^{2i\omega\tau} + c.c., \quad p_2 << p_1 \qquad (30)$$

with amplitudes

$$p_1(\sigma,\xi) = A(\sigma,\xi)e^{-i\omega^3\delta\, r_0\sigma} \qquad (31)$$

$$p_2(\sigma,\xi) = 2(ka)^2 e^{-8i\omega^3\delta\, r_0\sigma} \times$$
$$\int_0^\sigma exp\left(-\frac{2i\xi^2}{\sigma-\sigma'} + i6\omega^3\delta\, r_0\sigma'\right)\frac{d\sigma'}{\sigma-\sigma'} \times \qquad (32)$$
$$\times \int_0^\infty A^2(\sigma',\xi')J_0\left(\frac{4\xi\xi'}{\sigma-\sigma'}\right) exp\left(-\frac{2i\xi'^2}{\sigma-\sigma'}\right)\xi' d\xi'$$

$$A(\sigma,\xi) = \Pi\int_0^\infty J_1(\lambda)J_0(\lambda\xi)exp\left(\frac{i\lambda^2\sigma}{4}\right)d\lambda \qquad (33)$$

Π - amplitude of transducer normalised to p_f^0

The PASCAL program for calculation second harmonic in nondispersive medium with dissipation was modified for calculation rapidly oscillating integral (32). The approximation of the integrands by it's algorithm is more exact and converge more rapidly in comparison with one. As the illustration of the result of calculation we present a plot of $p_2(\sigma,\xi)$ in which one can see dispersion influence (Fig.1).

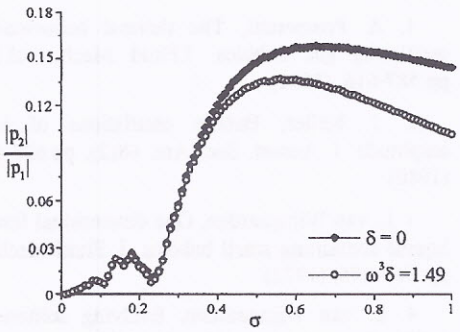

Fig.1 The amplitude of second harmonic as a function of axial distance.

The parameters we choose fit the range described in section 5. Those are $\alpha = 10^{-9}$, $R_0 = 10\mu m$, $f = 1{,}7\cdot 10^5\ Hz$, $a = 0.023\, m$.

6. Conclusions

Mixture we treat is compressible liquid with ideal gas bubbles which supposed to be weakly dispersive medium. Now the new system of equations for a compressible mixture dynamics in weakly nonlinear regime are written which leads to KdV (for a one-dimensional problem) and modified KZ (for three-dimensional one) equations. The mass concentration instead of volume one is used, that is more convenient because we don't need an additional equation of motion for this variable, since it is a constant. The new coefficients are obtained which related to the compressibility of a liquid. It is essential for small mass concentrations of gas ($\alpha < 10^{-8}$) in bubbly liquid. Moreover, our equation tends to those for pure liquid in a limit $\alpha \to 0$ unlike to equation which were obtained previously by other author. For sufficiently large α compressibility of liquid gives negligible contribution to coefficients and the equation corresponds to these ones obtained in [4] previously for incompressible liquid.

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