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Sound beams with shockwave pulses

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ABSTRACT

The beam equation for a sound beam in a diffusive medium, called the KZK (Khokhlov-Zabolotskaya-Kuznetsov) equation, has a class of solutions, which are power series in the transverse variable with the terms given by a solution of a generalized Burgers' equation. A free parameter in this generalized Burgers' equation can be chosen so that the equation describes an N-wave which does not decay. If the beam source has the form of a spherical cap, then a beam with a preserved shock can be prepared. This is done by satisfying an inequality containing the spherical radius, the N-wave pulse duration, the N-wave pulse amplitude and the sound velocity in the fluid.

A nonlinear limited sound beam propagating in the x direction is described by the Khokhlov-Zabolotskaya-Kuznetsov or KZK equation [1], [2] for the fluid velocity v:

$$\frac{\partial}{\partial\tau}\{\frac{\partial v}{\partial x} - \frac{\gamma+1}{2c_0^2}v\frac{\partial v}{\partial\tau} - \frac{b}{2c_0^3\rho_0}\frac{\partial^2 v}{\partial\tau^2}\} = \frac{c_0}{2}(\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}). \qquad (1)$$

The notation used is:

c_0 = sound velocity of the undisturbed fluid.

$\tau = t - \frac{x}{c_0}$.

$\gamma = \frac{c_p}{c_v}$, i.e. ratio between heat capacities of the fluid.

$b = \kappa(\frac{1}{c_p} + \frac{1}{c_v}) + \zeta + \frac{4}{3}\eta$, where κ is the heat conduction coefficient and ζ, η are the viscosities.

ρ_0 = density of the undisturbed fluid.

The beam is assumed to be cylindrically symmetrical. We make the substitutions

$$\rho^2 = y^2 + z^2 \qquad (2)$$

$$\beta = \frac{1}{2}(\gamma+1) \qquad (3)$$

and introduce the dimensionless variables V, θ, X, R according to the formulas

$$V = \frac{v}{v_0} \qquad (4)$$

$$\theta = \omega\tau \qquad (5)$$

$$X - X_0 = \frac{\beta\omega v_0}{c_0^2}x \qquad (6)$$

$$R = \frac{\rho}{a}, \qquad (7)$$

where a is the beam radius at the beam source, situated at $x = 0$.

Insertion of (2)-(8) into (1) gives the dimensionless beam equation

$$\frac{\partial}{\partial\theta}(\frac{\partial V}{\partial X} - V\frac{\partial V}{\partial\theta} - \epsilon\frac{\partial^2 V}{\partial\theta^2}) = \frac{N}{4}(\frac{\partial^2 V}{\partial R^2} + \frac{1}{R}\frac{\partial V}{\partial R}). \qquad (8)$$

The two dimensionless constants ϵ and N are given as

$$\epsilon = \frac{b\omega}{2\beta c_0 v_0\rho_0} \qquad (9)$$

$$N = \frac{2c_0^3}{\beta a^2 \omega^2 v_0}. \quad (10)$$

The parameter ϵ is assumed to be considerably less than unity.

With the substitution

$$R^2 = \frac{N}{2} r^2 \quad (11)$$

the circular symmetric beam equation (8) has the power series solution

$$\begin{aligned} V(X,\theta,r^2) = & V_0(X,\theta) + k\tfrac{r^2}{2X} V_{0\theta}(X,\theta) \\ & + \tfrac{1}{2!}\tfrac{k(k-1)}{2}\tfrac{r^4}{4X^2} V_{0\theta\theta}(X,\theta) \\ & + \tfrac{1}{3!} k r^6 \{ \tfrac{(k-1)(k-2)}{3!}\tfrac{1}{8X^3} V_{0\theta\theta\theta}(X,\theta) \\ & - \tfrac{k+1}{3}\tfrac{1}{8X^2}(V_{0\theta}V_{0\theta\theta})_\theta \} \\ & + \tfrac{1}{4!} k r^8 \{ \tfrac{(k-1)(k-2)(k-3)}{4!}\tfrac{1}{16X^4} V_{0\theta\theta\theta}(X,\theta) \\ & + \tfrac{k+1}{12}\tfrac{1}{16X^2}[-(V_{0\theta}V_{0\theta\theta})_{\theta\theta X} \\ & -(3k-5)\tfrac{1}{X}(V_{0\theta}V_{0\theta\theta})_{\theta\theta} \\ & +(V_0(V_{0\theta}V_{0\theta\theta})_\theta)_{\theta\theta} \\ & +\epsilon(V_{0\theta}V_{0\theta\theta})_{\theta\theta\theta\theta}]\} + \ldots \end{aligned} \quad (12)$$

The function $V_0(X,\theta)$ is a solution of the generalized Burgers' equation

$$V_{0X} - \frac{k}{X}V_0 - V_0 V_{0\theta} - \epsilon V_{0\theta\theta} = 0. \quad (13)$$

The number k is arbitrary. Terms of the order r^{2n+2} in the series (12) are calculated from terms of the order r^{2n} using the fact that terms of the order r^{2n+2} in V_0 at the right-hand side of (8) are compensated by terms of at most the order r^{2n} in V_0 at the lefthand side of (8). The special case $k = -1$, leading to the expansion (20), is that treated by Sionoid [3].

For $k \neq -1$ the equation (13) will be transformed. Using the substitutions

$$W = -X^{-k} V_0 \quad (14)$$

$$\xi = \frac{1}{k+1} X^{k+1} \quad (15)$$

in (13) we obtain

$$W_\xi + W W_\theta - \epsilon\{(k+1)\xi\}^{-\frac{k}{k+1}} W_{\theta\theta} = 0. \quad (16)$$

For arbitrary $k \neq -1$ the wave equation (16) formally describes plane waves propagating in a medium with variable viscosity.

For $k = 1$ the following equation is obtained from (16):

$$W_\xi + W W_\theta - \frac{\epsilon}{\sqrt{(2\xi)}} W_{\theta\theta} = 0. \quad (17)$$

Generalized Burgers' equations of the type

$$W_\xi + W W_\theta - \epsilon G(\xi) W_{\theta\theta} = 0 \quad (18)$$

are treated by Crighton and Scott [4]. They show that the N-wave boundary condition

$$\begin{aligned} W(1,\theta) &= \theta,\ |\theta| < 1 \\ W(1,\theta) &= 0,\ |\theta| > 1 \end{aligned} \quad (19)$$

leads to the "outer" solution

$$\begin{aligned} W^{(e)}(\xi,\theta) &= \tfrac{\theta}{\xi} + o(\epsilon^n),\ |\theta| < \sqrt{\xi} \\ W^{(e)}(\xi,\theta) &= 0,\ |\theta| > \sqrt{\xi}. \end{aligned} \quad (20)$$

An "inner" solution to (18) in the neighborhood of $\theta = \sqrt{\xi}$ is found by asymptotic matching [4]. For $G(\xi) = (2\xi)^{-\frac{1}{2}}$ this solution is

$$\begin{aligned} W^i(\xi,\theta^*) = & \tfrac{1}{2\sqrt{\xi}}\{1 - \tanh \tfrac{\theta^* - \sqrt{2}(1-\xi^{\frac{1}{2}})}{2\sqrt{2}}\} \\ + & \epsilon W_1^i + \ldots, \end{aligned} \quad (21)$$

where

$$\theta^* = \frac{\theta - \sqrt{\xi}}{\epsilon}. \quad (22)$$

It can be shown that the coefficient $\frac{\epsilon}{\sqrt{(2\xi)}}$ of the second derivative term in (17) has exactly the necessary decreasing behaviour with ξ for giving a nongrowing shockwidth, which according to (21) is $2\sqrt{2}\epsilon$.

A boundary condition for the KZK equation (1) or (8) can be prepared in order to give a preserved shock solution. This shock solution will be found by a solution of (17) used in the expansion (12) with an appropriate value of k. The original N-wave is generated on a spherical concave cap, whose surface has the equation

$$(x-d)^2 + y^2 + z^2 = d^2. \quad (23)$$

The cap surface is that part of this spherical surface which fulfils the inequality

$$y^2 + z^2 \leq a^2. \quad (24)$$

Assuming that the spherical radius d of the cap is much greater its its intersection radius,

$$d >> a, \quad (25)$$

the cap equation is approximated:

$$x \approx \frac{y^2 + z^2}{2d}. \quad (26)$$

Following Ystad and Berntsen [5] we formulate a boundary condition on the plane $x = 0$ equivalent to the boundary condition on the curved cap surface. We assume a boundary condition in which the wave phase is constant on the cap surface and the wave amplitude depends only on the distance from the beam axis. The equivalent boundary condition at $x = 0$ then becomes:

$$\begin{aligned} v(x=0,\rho,t) &= v_0 g(\tfrac{\rho^2}{a^2}) F(\omega t + \tfrac{1}{2}\tfrac{\omega \rho^2}{c_0 d}),\ \rho < a \\ v(x=0,\rho,t) &= 0,\ \rho > a, \end{aligned} \quad (27)$$

where (2) is used. The function g is constant for $\rho < a$ and zero for $\rho > a$.

The function F in the equivalent boundary condition (27) now has to be an N-wave:

$$\begin{aligned} F(\theta) &= -\theta,\ |\theta| < 1 \\ F(\theta) &= 0,\ |\theta| > 1. \end{aligned} \quad (28)$$

It can be shown that, for $k > 0$, the boundary condition (27) is compatible with the series solution (12) if

$$d = \frac{(k+1)c_0^2 T}{k\beta u_0}, \quad (29)$$

where $T = \omega^{-1}$. The result (29) is the main result of this investigation. The lowest k value, for which the shock width does not grow during the propagation of the N-wave, is $k = 1$. An inequality, which has to be fulfilled for an N-wave, created on a spherical cap with radius d, of duration $2T$ and amplitude u_0 thus is

$$d \leq \frac{2c_0^2 T}{\beta u_0}, \quad (30)$$

if the shock shall not decay during the propagation of the wave.

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