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The saddle point method applied to some problems in acoustics

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ABSTRACT

The paper presents examples of applying the saddle point method (called also the method of the steepest descent path) to some problems of acoustics. The method is shortly reminded in its basic form and more complex options which, however, improve the results and widen the range of eventual applications. As examples the phenomena of reflection and transmission of the spherical wave at a fluid-fluid interface and the far field radiated from the circular duct are discussed.

INTRODUCTION

In the process of mathematical description of a physical phenomena we often apply sophisticated mathematical methods as integral transforms, integral representation etc, which may darken physical interpretation of the results obtained.

The aim of the paper is to present the saddle point method as an useful tool to solve physical problems. The examples (reflection/ transmission of sound through a plane interface separating two fluids, far field radiated from cylindrical duct), all of considerable theoretical and practical meaning, have been chosen to demonstrate the large spectrum of advantages from applying the saddle point method, which not only simplifies mathematical formulae but also allows for clear and meaningful physical interpretation.

THE SADDLE POINT METHOD

The saddle point method is used to calculate approximately contour integrals of the type [1]

$$I(\lambda) = \int_C G(z)e^{\lambda g(z)}dz, \qquad (1)$$

in the complex plane z, where λ is real parameter and $G(z)$, $g(z)$ - analytic functions. The method consists of three main steps:

1^o finding one or more saddle points, defined by the criterion

$$g'(z) = 0 \quad \text{at} \quad z = \xi\,;$$

2^0 deforming, respecting all necessary rules, the contour of integration into the steepest descent path, which is defined as the path in the complex plane that passes through the saddle point

ξ and along which the real part of $g(z)$ ($\text{Re } g(z) = g_R$) decreases most rapidly. By a property of analytic functions $\text{Im } g(z) = g_I$ remains constant on this contour.

The steepest descent path can be defined by means of a real parameter s

$$g(z) = g(\xi) - s^2, \tag{2}$$

(s^2 because at each point $z \neq \xi$, $g(z) < g(\xi)$, what ensures real s). Equating the last formula with, limited to the second derivative, Taylor series expansion for $g(z)$ about the saddle point ξ: $g(z) = g(\xi) + g'(\xi)(z-\xi) + \frac{1}{2}g''(\xi)(z-\xi)^2$ and taking into account $g'(\xi) = 0$ we can express the parameter s by variable z

$$s = \sqrt{-g''(\xi)/2}(z - \xi). \tag{3}$$

Denoting the beginning and the end of the contour C by s_1 and s_2 we obtain

$$I(\lambda) = (-2/g''(\xi))^{\frac{1}{2}} e^{i\lambda g_I(\xi)} \int_{s_1}^{s_2} e^{\lambda(g_R(\xi)-s^2)} G(s) ds; \tag{4}$$

3^0 performing integration. The integrand, especially for large positive λ, is small everywhere except the vicinity of the saddle point, ξ, ($s = 0$). For slowly varying function $G(s)$, assume $G(s) \cong G(s=0) = G(\xi)$and extend the interval into $(-\infty, +\infty)$. As a result the Gauss integral $\int_{-\infty}^{+\infty} e^{-\lambda s^2} ds = \sqrt{\pi/\lambda}$ appears. Finally, we can write

$$I(\lambda) = \sqrt{\frac{-2\pi}{\lambda g''(\xi)}} e^{\lambda g(\xi)} G(\xi). \tag{5}$$

The formula (5) is often called the first order saddle point approximation. If we deal with integrals strictly fulfilling the above listed conditions we can easily evaluate the integral. Unfortunately, we hardly come across such an ideal situation in practice. On the contrary - usually many difficulties arise, as multi-valued functions, for which we have to construct Riemann's surface and on which we have to consider not only the singularities but also the branch points and branch cuts. If we want to deform a contour of integration on the Riemann surface we have to take precautions - the beginning and the end of the path must lie on the same leaf. In acoustical problems we often deal with double - valued functions (containing square roots) and properly deformed contour of integration must cross the cut lines zero or even number of times. Quite often λ parameter is not large everywhere or $G(z)$ is not slowly varying function near the saddle point. For these reasons some improvements of the basic method are introduced, three of which will be discussed below.

Inclusion of the argument of the G(z) function into the exponential function.

This is the easiest way of improving the approximation and it is quite efficient in some cases (as an example we will discuss the case of reflection of waves at the incident angles larger than the critical, where the reflection coefficient is a complex number of moduli 1). Using the relation $G(z) = |G(z)| \exp(i\Gamma)$ we obtain the function in the exponent in the form

$$g_1(z) = g(z) + (i/\lambda)\Gamma(z), \tag{6}$$

and the saddle point criterion

$$g'(\xi) + (i/\lambda)\Gamma'(\xi) = 0, \tag{7}$$

from which we obtain the integral in the form

$$I(\lambda) = \sqrt{\frac{-2\pi}{\lambda g''(\xi) + i\Gamma''(\xi)}} e^{\lambda g(\xi)} G(\xi). \tag{8}$$

In this approximation the phase of the function $G(z)$ affects the saddle point location and the steepest descent path. This variant of the method is suitable especially for functions of steady amplitude and varying argument.

Inclusion of the G(z) function into the exponential function.

In this we take one step further by allowing both - the amplitude and phase of the $G(z)$ function to influence on the saddle point location and the steepest descent path writing $G(z) = \ln(\exp G(z))$. This variant is especially useful when the $G(z)$ function is not varying slowly and can be compared with the exponential function. Incorporation of the $G(z)$ into the exponential function leads to the following form of the exponent [6]

$$g_2(z) = g(z) + (1/\lambda)\ln G(z), \qquad (9)$$

and the saddle point criterion

$$g'(\xi) + \frac{1}{\lambda}\frac{G'(\xi)}{G(\xi)} = 0 \,, \qquad (10)$$

which results in the formula for the integral:

$$I(\lambda) = \sqrt{\frac{-2\pi}{\lambda g'' + \frac{GG''-(G')^2}{G^2}}}G(\xi)e^{\lambda g(\xi)} \,, \qquad (11)$$

where for simplicity it was not indicated that all terms are taken in the saddle point $z = \xi$. From the above we see that the saddle points are, in general, complex.

Second and further approximations.

As was told before the basic formula (5) is often called the first order approximation. To obtain the second one we have to expand the integrand $G(z)$ into the Taylor series in the neighbourhood of the saddle point ξ (what is equivalent with expanding $G(s)$ in the neighbourhood of $s = 0$): $G(s) \cong G(0) + G'(0)s + \frac{1}{2}G''(0)s^2$, what leads to

$$I(\lambda) = \sqrt{\frac{-2}{g''(\xi)}}e^{\lambda g(\xi)} \times$$

$$\times \left[\sqrt{\frac{\pi}{\lambda}}G(\xi) + \frac{1}{2}\left(\frac{-2}{g''(\xi)}\right)G''(\xi)\frac{\pi}{2\lambda}\right]. \qquad (12)$$

It is worth mentioning that the formulae derived in this section are valid under the assumption that in the saddle point ξ the second derivative $g''(\xi) \neq 0$, otherwise the obtained integrals become infinite. The case $g'(\xi) = 0$, $g''(\xi) = 0$ cannot be treated by means of this method.

PLANE-WAVES REPRESENTATION OF THE SPHERICAL WAVE

In the following we will discuss the problem of the sound field at a plane fluid-fluid interface. The field is generated by a point source located in the upper half plane the lower medium is of the higher velocity (air/water, water/sand), what results in the phenomena of total reflection above the critical angle. One of the consequences is appearance of a lateral wave, well known in shallow water acoustics and seismology.

The first step is to present the plane wave integral representation of a spherical wave.

Assume a point, sound source of monochromatic wave, which acoustic potential at a distance R from the source is $\phi_0(R) = e^{ikR}/R$, where $k = \omega/c$ is the wavenumber, with ω denoting the wave frequency and c the speed of sound. In the Cartesian co-ordinates (x, y, z) we can mathematically express the potential of a spherical wave as a surface integral [1]

$$\phi_0 = \frac{e^{ikR}}{R} = \int\int_{-\infty}^{+\infty} \frac{e^{i(k_x x + k_y y + k_z z}}{2\pi k_z} dk_x dk_y, \qquad (13)$$

where $R^2 = x^2 + y^2 + z^2$ and $k^2 = k_x^2 + k_y^2 + k_z^2$. The integration is performed over the entire plane and we allow for imaginary values of k_z. In the spherical co-ordinates we can write Eq. 13 in a form:

$$\frac{e^{ikR}}{R} = ik \int_0^{\frac{\pi}{2}-i\infty} J_0(kr\sin\theta)e^{ikR\cos\theta\cos\theta_0}d\theta, \qquad (14)$$

where $r = R\sin\theta_0$, $J_0()$ represents the Bessel function of order zero. As $k_z = k\cos\theta$ for real k_z changing in the limits $0 \le k_z \le k$ we have real values of θ changing from $\pi/2$ to 0. For imaginary k_z, with increasing positive imaginary part we obtain the range of θ from $\frac{\pi}{2}$ to $\frac{\pi}{2} - i\infty$ (we choose $-i\infty$ for physical reasons).

The potential of a spherical wave is possibly expressed by means of the Hankel function $H_0^{(1)}$. There are two main reasons for which we will stick to that form - the symmetry properties of the new contour of integration and the simplicity of asymptotic form of the Hankel function. Applying the identity $J_0(z) = \frac{1}{2}[H_0^{(1)}(z) - H_0^{(1)}(-z)]$ and introducing under the integral the asymptotic form for the Hankel's function for $|z| \gg 1$,

$$H_0^{(1)}(z) = (\frac{2}{\pi z})^{\frac{1}{2}} \exp i(z - \frac{\pi}{4})(1 + \frac{1}{8iz} + \cdots) \cong$$

$$\cong \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \exp i(z - \frac{\pi}{4}), \qquad (15)$$

we obtain in the first approximation

$$\frac{e^{ikR}}{R} = c \int_{-\frac{\pi}{2}+i\infty}^{\frac{\pi}{2}-i\infty} e^{ikR\cos(\theta-\theta_0)} \sqrt{\sin\theta} d\theta\,, \quad (16)$$

where $c = \sqrt{ik/2\pi R \sin\theta_0}$. In the last formula we can easily recognise the form suitable for applying the saddle point method. The spherical wave is decomposed into an infinite number of individual plane waves incident at the interface at an angle θ. Note that to each of these contributing plane waves we can apply the simple rules of reflection and transmission (Snell's law).

FLUID - FLUID INTERFACE

In the following we will consider the reflection /transmission of a spherical wave at a fluid - fluid boundary [2, 3, 6].

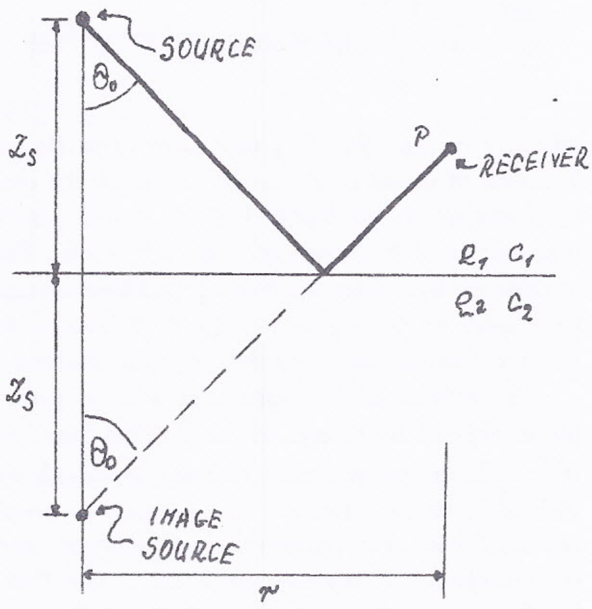

Fig. 1. Geometry of the reflected field.

The point source (Fig. 1) is located at the height z_s in a medium characterised by the density ϱ_1 and the speed sound c_1, which in the lower medium are equal to ϱ_2, c_2, respectively. We assume $c_2 > c_1$. The plane wave reflection coefficient is [4]

$$V(\theta) = \frac{m\cos\theta - \sqrt{n^2 - \sin^2\theta}}{m\cos\theta + \sqrt{n^2 - \sin^2\theta}}, \qquad (17)$$

where $m = \varrho_2/\varrho_1$ and $n = c_1/c_2$. It is useful to remind that in the considered case $n < 1$, so the critical angle, at which the total reflection appears, is defined as $\theta_{cr} = \arcsin(n)$.As long as we deal with the homogeneous media n and θ_{cr} are real. We can introduce the attenuation constant by including an imaginary part to the sound speed, what results in complex n, with a positive imaginary part. At the moment let us focus on the case of real n and θ_{cr}. Applying Eq. (16) to the reflection phenomena we can write

$$\phi_{ref}(R, \theta_0) = c \int_{-\frac{\pi}{2}+i\infty}^{\frac{\pi}{2}-i\infty} e^{ikR\cos(\theta-\theta_0)} V(\theta) \sqrt{\sin\theta} d\theta, \qquad (18)$$

which is the basic formula for further investigations. Note that apart from using the asymptotic form of the Hankel function no approximations were made till now.

Comparing (18) with (1), according to the basic form of the saddle point method we obtain

$$\lambda = kR \;, \quad g = i\cos(\theta - \theta_0) \;, \quad G = V(\theta)\sqrt{\sin\theta}, \qquad (19)$$

and the saddle point occurs at $\xi = \theta_0$.

We have to check if the $V(\theta)$ function fulfils the required conditions. If we regard angles of incidence $\theta < \theta_{cr}$, $V(\theta)$ is real and slowly varying function, thus

$$\phi_{ref} = V(\theta_0)\frac{e^{ikR}}{R} \quad , \; \theta_0 < \theta_{cr}. \qquad (20)$$

A question arises if the last formula is fulfilled for the angles of incidence larger than (although not very close to) the critical angle.

The steepest descent path must be obtained by a smooth deformation of the initial contour, so for $\theta_0 > \theta_{cr}$ it would cross the branch cut (Fig. 2.).

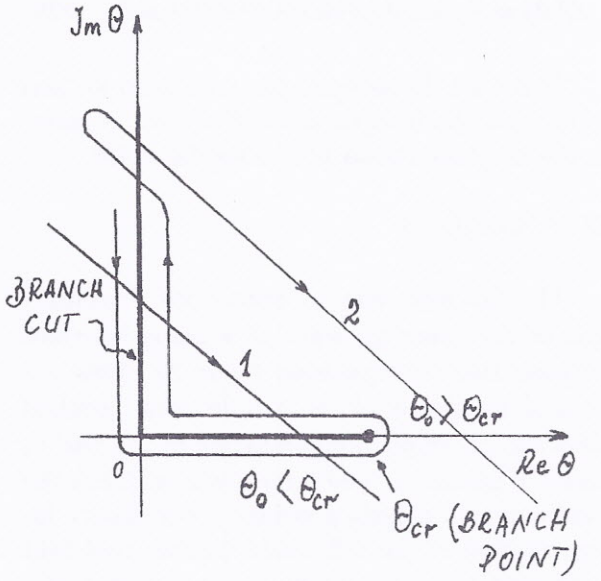

Fig. 2. The path of integration for different value of the angle of incidence: curve 1 for $\theta < \theta_{cr}$, 2 for $\theta > \theta_{cr}$.

In the first approximation, starting from (18) we obtain, for $\theta_0 > \theta_{cr}$

$$\phi_{ref}(\theta_0, R) = \int\limits_{K+L} = V(\theta_0)\frac{e^{ikR}}{R} +$$

$$+ c \int\limits_{L} e^{ikR\cos(\theta\theta_0)} V(\theta)\sqrt{\sin\theta}\, d\theta, \qquad (21)$$

where L denotes the contour along the branch cut. The first integral represents geometric (Snell's) wave, the second - lateral wave (Fig. 3).

$$\phi_{ref} = \underset{\theta<\theta_0}{\phi_{Snell}} + \underset{\theta>\theta_0}{\phi_{lateral}}. \qquad (22)$$

Moreover, for $\theta > \theta_0$ the $V(\theta)$ function becomes complex $V(\theta) = |V|e^{i\psi(\theta)}$. Including the argument $\psi(\theta)$ into the exponential function we obtain [6]

$$g = i\cos(\theta - \theta_0) + \frac{i}{kR}\psi(\theta) \ , \ G = |V|\sqrt{\sin\theta}, \qquad (23)$$

while in the most exact variant, when amplitude and phase of the reflection coefficient influence on the saddle point location, we write $V(\theta) = \exp(\ln V(\theta))$, so

$$g = i\cos(\theta - \theta_0) + \frac{i}{kR}\ln V(\theta) \ , \ G = \sqrt{\sin\theta}. \qquad (24)$$

This method is especially suitable for rapidly changing $V(\theta)$, especially in the vicinity of the saddle points, which, unfortunately, could be derived only by means of numerical computations. They are complex and could appear in pairs what additionally complicates the steepest descent path [2, 6].

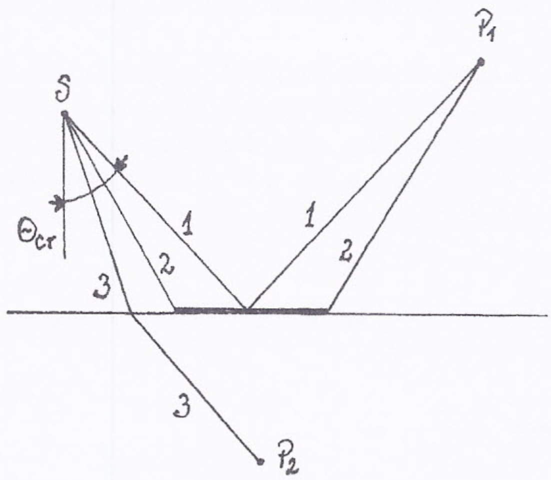

Fig. 3. Contribution of different waves at the observation point in the upper (P_1) and lower (P_2) medium; 1 - totally reflected wave, 2 - lateral wave, 3 - transmitted wave.

Anyhow, independently of the steepest descent path method we are following, for angles of incidence exceeding the critical angle, $\theta_0 > \theta_{cr}$, we obtain the reflected field composed of two factors - the wave reflected along the Snell's law and the lateral wave.

THE FAR FIELD OF A CIRCULAR DUCT

Considering harmonic vibrations, the l-th mode acoustical potential of a semi-infinite cylindrical duct of radius a, with the outlet at $z = 0$

takes, in the cylindrical co-ordinates, the form [5]:

$$\phi_l(\varrho, z) = \frac{ai}{4} \int\limits_C dw \exp(iwz) v F_l(w) \times$$

$$\times \begin{cases} H_0^{(1)}(v\varrho) J_1(va); & \varrho > a \\ H_1^{(1)}(va) J_0(v\varrho); & \varrho < a \end{cases}, \qquad (25)$$

where w, v are the axial and radial wavenumbers, respectively, and $w^2 + v^2 = k^2$, $F_l(w)$ is the Fourier transform of the discontinuity (jump) of the potential on duct's wall. The contour of integration consists of the real axis and the loop around the point $w = -\gamma_l$, which is the only singularity point of the function $F_l(w)$ [5].

To apply the saddle point method to calculate the acoustic field outside the wave - guide we introduce the complex variable α as $w = k \sin \alpha$ and make use of the asymptotic form of Hankel's function (15) what leads to

$$\phi_l(R, \vartheta) = c e^{\frac{i\pi}{4}} \int\limits_{C_1} W_l(\theta, \vartheta) e^{ikz \sin(\theta+\vartheta)} \sqrt{\cos\theta} d\theta, \qquad (26)$$

where

$$W_l(\theta, \vartheta) = J_1(ka \cos\theta) F_l(k \sin\theta) \times$$

$$\times \cos\theta \left(1 + \frac{1}{8ikR \sin\vartheta \cos\theta}\right), \qquad (27)$$

and c =const. The contour C_1 starts at $\pi/2 - i\infty$, passes near $\pi/2$, cuts the real axis at 0 and, near $-\pi/2$, goes asymptotically to $-\pi/2 + i\infty$, surrounding in the meanwhile the point $-\theta_l = \arcsin(-\gamma_l/k)$.

Following the rules of the steepest descent path method we come to the result:

$$\phi_l(R, \vartheta) = \frac{ak}{2} \sin\vartheta J_1(ka \sin\vartheta) F_l(k \cos\vartheta) \frac{e^{ikR}}{R}. \qquad (28)$$

To recapitulate, in the first approximate we have obtained the potential in the form of the spherical wave multiplied by the directivity function

$$d_l(\vartheta) = \frac{ak}{2} \sin\vartheta J_1(ka \sin\vartheta) F_l(k \cos\vartheta). \qquad (29)$$

To obtain the second approximation we have to expand the integrand $W_l(\theta, \vartheta)$ into a series in the neighbourhood of the saddle point.

CONCLUSIONS

The far field due to reflection/ transmission of the spherical wave at a plane interface between two homogeneous fluids has been examined by means of the saddle point method allowing for simple interpretation of the field in terms of geometric and lateral waves. Thus the saddle point analysis provides a ray theory interpretation of the reflected/ transmitted field . Applied to the problem of the far field of a circular duct results in obtaining the formula for the directivity function. It seems that the method allows for simple physical interpretation of the results obtained.

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